Hyperbolicity of almost periodic evolution families

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Abstract

We characterize hyperbolicity of periodic and certain asymptotically almost periodic evolution families $\mathcal{U} = \{U(t,s) : t \geq s\}$ and establish conditions on $\mathcal{U}$ which lead to unique periodic resp. asymptotically almost periodic mild solutions $u(t) = U(t,s)u(s) + \int_s^t U(t,\tau)f(\tau)d\tau,\ t \geq s \in \mathbb{R}$, provided that f is periodic resp. asymptotically almost periodic with relatively compact range. Using the developed methods, we also discuss weak almost periodicity in the sense of Eberlein for hyperbolic evolution families.

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1 Introduction

The purpose of this paper is to characterize hyperbolicity of periodic evolution families and to obtain similar results for asymptotically almost periodic evolution families. Here, $\mathcal{U} = \{U(t,s) : t \geq s\}$ is called an *evolution family* in the space $\mathcal{L}(X)$ of bounded linear operators on a Banach space X if

$$U(t,t) = Id,\ U(t,r)U(r,s) = U(t,s) \text{ for } t \geq r \geq s \in \mathbb{R},$$

$$\{(t,s) \in \mathbb{R}^2 : t \geq s\} \to \mathcal{L}(X) :\ (t,s) \mapsto U(t,s) \text{ is strongly continuous},$$

$$\text{there are constants } M \geq 1,\ \omega \in \mathbb{R} \text{ such that } \|U(t,s)\| \leq Me^{\omega(t-s)},\ t \geq s \in \mathbb{R}.$$

For a suitable function $f : \mathbb{R} \to X$, we consider the "variation of constants formula"

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\tau)f(\tau)d\tau,\ t \geq s. \tag{1}$$

In case the evolution family $\mathcal{U}$ is induced by the solutions of a non-autonomous Cauchy problem

$$\dot{u}(t) = A(t)u(t),\ t \in \mathbb{R}, \tag{2}$$

a function u satisfying (1) corresponds to a mild solution of the inhomogeneous equation

$$\dot{u}(t) = A(t)u(t) + f(t), \ t \in \mathbb{R}. \tag{3}$$

Conditions on the operators $A(t)$, $t \in \mathbb{R}$, such that the solutions of (2) define an evolution family $\mathcal{U}$ on X can be found e.g. in [Fat83, Paz83, Tan79].

We shall be concerned here with evolution families according to the above definition, without assuming the existence of a related Cauchy problem. If $\mathcal{F}(\mathbb{R}, X)$ is a suitable Banach space of functions from $\mathbb{R}$ to X, then, under certain conditions,

$$T(t)f := U(\cdot, \cdot - t)f(\cdot - t), \ t \geq 0, \ f \in \mathcal{F}(\mathbb{R}, X)$$

defines a $\mathcal{C}_0$-semigroup $\mathcal{T} = \{T(t) : t \geq 0\}$ on $\mathcal{F}(\mathbb{R}, X)$, the so called *evolution semigroup* (see [How74, Nag95, Rau94, RRS96, RS96b] and the literature cited therein). Let $Q := Id - P$ for a bounded projection P on a Banach space X. Denote by T_Q and T_P the restriction of $T \in \mathcal{L}(X)$ to QX and PX, respectively. Recall that a $\mathcal{C}_0$-semigroup $\mathcal{T}$ on X is called *hyperbolic* if there exists a bounded projection P such that, for $t \geq 0$,

(i) $PT(t) = T(t)P$,

(ii) the operator $T_Q(t)$ is invertible,

(iii) there are constants $N \geq 1$ and $\alpha > 0$ such that

$$\|T_P(t)P\| \leq Ne^{-\alpha t},$$
$$\|T_Q(t)^{-1}Q\| \leq Ne^{-\alpha t}.$$

Hyperbolicity of a $\mathcal{C}_0$-semigroup is characterized by the condition $\Gamma \subseteq \rho(T(t_0))$ for one (hence, for all) $t_0 > 0$, where Γ denotes the unit circle in $\mathbb{C}$ and $\rho(T(t_0))$ is the resolvent set of $T(t_0)$ [Nag86, A-III.3]. Hyperbolicity of an evolution semigroup $\mathcal{T}_0$ on $C_0(\mathbb{R}, X)$ (the Banach space of bounded, continuous functions vanishing at $\pm\infty$) has been considered in [Rau94, LMS96, LMS94, LR95, RS94, RS96b]. For evolution families, we make the following definition (see [LMS96] and [RS96b, Section 4]).

Definition 1.1 An evolution family $\mathcal{U}$ is called *hyperbolic* if there are projections $P(t)$ such that $P(\cdot) \in C_b(\mathbb{R}, \mathcal{L}_s(X))$ (i.e., a bounded, continuous function from $\mathbb{R}$ to $\mathcal{L}(X)$ equipped with the strong operator topology) and for $t \geq s$,

(i) $P(t)U(t,s) = U(t,s)P(s)$,

(ii) the restriction $U_Q(t,s) : Q(s)X \to Q(t)X$ is invertible,

(iii) there are constants $N \geq 1$ and $\alpha > 0$ such that

$$\begin{aligned} \|U_P(t,s)P(s)\| &\leq Ne^{-\alpha(t-s)}, \\ \|U_Q(t,s)^{-1}Q(t)\| &\leq Ne^{-\alpha(t-s)}. \end{aligned}$$

Our main objective is the investigation of hyperbolic almost periodic evolution families. Therefore, let us recall some of the notions of almost periodicity which shall come into play later on.

Definition 1.2 A bounded, continuous function $f \in C_b(\mathbb{R}, X)$ is said to be *q-periodic (q-p.)* $(0 < q \in \mathbb{R})$ if $f(t+q) = f(t)$ for all $t \in \mathbb{R}$. The Banach space of q-p. functions will be denoted by $P_q(\mathbb{R}, X)$.

A bounded, continuous function $f \in C_b(\mathbb{R}, X)$ is said to be *almost periodic* (a.p.) if the set of translates $\{f_\omega := f(\cdot + \omega) : \omega \in \mathbb{R}\}$ is relatively compact in the Banach space $(C_b(\mathbb{R}, X), \|\cdot\|_\infty)$ or, equivalently, if, given any $\epsilon > 0$, there exists a relatively dense subset $P(\epsilon)$ in $\mathbb{R}$ such that $\|f(t+\tau) - f(t)\| < \epsilon$ for each *ϵ-almost period* $\tau \in P(\epsilon)$ and every $t \in \mathbb{R}$. The Banach space of a.p. functions will be denoted by $AP(\mathbb{R}, X)$ [Boh32, Boc33, Cor68].

A function $f : \mathbb{R} \to X$ is said to be *asymptotically almost periodic* (a.a.p.) if there is an a.p. function g and a function

$$h \in C_0^+(\mathbb{R}, X) := \{f : \mathbb{R} \to X : f \text{ is bounded, uniformly continuous, and } \lim_{t\to\infty} \|h(t)\| = 0\}$$

such that $f = g + h$. The Banach space of a.a.p. functions will be denoted by $AAP^+(\mathbb{R}, X)$ [Fré41, RS88, Zai85, RV95]. We shall write *a.a.p.r. function* for a function of the Banach space

$$AAP_r^+(\mathbb{R}, X) := \{f \in AAP^+(\mathbb{R}, X) : f \text{ has relatively compact range}\}.$$

An operator-valued function $P(\cdot) : \mathbb{R} \to \mathcal{L}(X)$ is called q-p. resp. a.a.p.r. if $P(\cdot)x$ is q-p. resp. a.a.p.r. for all $x \in X$. The Banach space of q-p. resp. a.a.p.r. operator-valued functions will be denoted by $P_q(\mathbb{R}, \mathcal{L}_s(X))$ resp. $AAP_r^+(\mathbb{R}, \mathcal{L}_s(X))$.

Our interest here is directed to q-p. resp. a.a.p.r. evolution families $\mathcal{U}$, i.e.,

$$U(t + \cdot, s + \cdot)x : \mathbb{R} \to X$$

are q-p. resp. a.a.p.r. functions for all $t \geq s$ and $x \in X$. In Section 2, we study multiplication operators on spaces of almost periodic functions. Then, in Section 3, we introduce evolution semigroups for almost periodic evolution families and establish spectral properties which will be useful in our discussion of hyperbolicity in the fourth section. We obtain characterizations of hyperbolic q-p. evolution families and corresponding results for a.a.p.r. evolution families. In Section 5, we turn to existence and uniqueness of q-p. and a.a.p.r. solutions of the integral equation (1). In the final section, we establish hyperbolicity of evolution families which are weakly almost periodic in the sense of Eberlein.

Let us fix some notations. For a linear operator A, denote by $A\sigma(A)$ resp. $\sigma(A)$ the approximate point spectrum resp. the spectrum of A. For the function constant $x \in X$ we also write $\mathbf{1} \otimes x$.

2 Multiplication operators on spaces of almost periodic functions

In this section, we are concerned with multiplication operators on the space of q-p. resp. a.a.p.r. functions (cf. [Gra97] for multiplication operators on $C_0(\mathbb{R}, X)$).

Definition 2.1 *(i)* An operator $T \in \mathcal{L}(P_q(\mathbb{R}, X))$ is called a *multiplication operator on* $P_q(\mathbb{R}, X)$ if there exists a function $F : \mathbb{R} \to \mathcal{L}(X)$ such that $Tf(s) = F(s)f(s)$ for all $f \in P_q(\mathbb{R}, X)$ and $s \in \mathbb{R}$.

(ii) An operator $T \in \mathcal{L}(AAP_r^+(\mathbb{R}, X))$ is called a *multiplication operator on* $AAP_r^+(\mathbb{R}, X)$ if there exists a function $F : \mathbb{R} \to \mathcal{L}(X)$ such that $Tf(s) = F(s)f(s)$ for all $f \in AAP_r^+(\mathbb{R}, X)$ and $s \in \mathbb{R}$.

In order to characterize such operators, we need the following observations.

Lemma 2.2 *(i) Let $T \in \mathcal{L}(P_q(\mathbb{R}, X))$ and $F : \mathbb{R} \to \mathcal{L}(X)$ such that $Tf(s) = F(s)f(s)$ for all $s \in \mathbb{R}$ and $f \in P_q(\mathbb{R}, X)$. Then $F \in P_q(\mathbb{R}, \mathcal{L}_s(X))$.*

(ii) Let $T \in \mathcal{L}(AAP_r^+(\mathbb{R}, X))$ and $F : \mathbb{R} \to \mathcal{L}(X)$ such that $Tf(s) = F(s)f(s)$ for all $s \in \mathbb{R}$ and $f \in AAP_r^+(\mathbb{R}, X)$. Then $F \in AAP_r^+(\mathbb{R}, \mathcal{L}_s(X))$.

Proof. Note that $F(s)x = T\mathbf{1} \otimes x(s)$ for all $x \in X$ and $s \in \mathbb{R}$. □

Furthermore, we use an approximation property of periodic functions.

Lemma 2.3 *Let $f \in P_q(\mathbb{R}, X)$ and suppose that $f(s) = 0$ for some $s \in \mathbb{R}$. Then there exist functions $\varphi_n \in P_q(\mathbb{R})$, $n \in \mathbb{N}$, such that $\varphi_n(s) = 0$ and $\varphi_n f \to f$ in $C_b(\mathbb{R}, X)$ as $n \to \infty$.*

Proof. Let $s \in \mathbb{R}$ such that $f(s) = 0$ and choose functions $\varphi_n \in P_q(\mathbb{R}, X)$ such that

$$\varphi_n(t) = \begin{cases} 1 & : \quad t \in [s + \frac{1}{n}, s + q - \frac{1}{n}] \\ 0 & : \quad t = s \end{cases} , \; n \in \mathbb{N}.$$

This immediately yields the assertion. □

Keeping this considerations in mind, we obtain a description of multiplication operators on $P_q(\mathbb{R}, X)$ which will come into play later on.

Theorem 2.4 *Let $T \in \mathcal{L}(P_q(\mathbb{R}, X))$. Then the following assertions are equivalent.*

(i) The operator T is a multiplication operator on $P_q(\mathbb{R}, X)$.

(ii) $T\varphi f = \varphi T f$ for all $f \in P_q(\mathbb{R}, X)$ and $\varphi \in P_q(\mathbb{R})$.

(iii) If $f \in P_q(\mathbb{R}, X)$ such that $f(s) = 0$ for some $s \in \mathbb{R}$, then $Tf(s) = 0$.

Proof. (i)⇒(ii): Let $F \in P_q(\mathbb{R}, \mathcal{L}_s(X))$ be the function representing T. Then, for $f \in P_q(\mathbb{R}, X)$ and $\varphi \in P_q(\mathbb{R})$, it follows that

$$T\varphi f(s) = F(s)\varphi(s)f(s) = \varphi(s)F(s)f(s) = \varphi(s)Tf(s)$$

for all $s \in \mathbb{R}$.

(ii)⇒(iii): Let $f \in P_q(\mathbb{R}, X)$ such that $f(s) = 0$ for some $s \in \mathbb{R}$. Let φ_n, $n \in \mathbb{N}$, be given as in Lemma 2.3. Then

$$Tf(s) = \lim_{n\to\infty} T\varphi_n f(s) = \lim_{n\to\infty} \varphi_n(s)Tf(s) = 0.$$

(iii)⇒(i): Define $F : \mathbb{R} \to \mathcal{L}(X)$ by means of $F(s)x := Tf(s)$ for $f \in P_q(\mathbb{R}, X)$ satisfies $f(s) = x$. Note that (iii) implies that $F(s)x$ is well-defined. □

The a.a.p.r. version of this result can be obtained by similar arguments, using the approximation theorem for a.p. functions.

Theorem 2.5 *Let $T \in \mathcal{L}(AAP_r^+(\mathbb{R}, X))$. Then the following assertions are equivalent.*

(i) The operator T is a multiplication operator on $AAP_r^+(\mathbb{R}, X)$.

(ii) $T\varphi f = \varphi Tf$ for all $f \in AAP_r^+(\mathbb{R}, X)$ and $\varphi \in AAP^+(\mathbb{R})$.

(iii) If $f \in AAP_r^+(\mathbb{R}, X)$ such that $f(s) = 0$ for some $s \in \mathbb{R}$, then $Tf(s) = 0$.

Proof. The implications (i)⇒(ii) and (iii)⇒(i) can be shown as the corresponding implications in Theorem 2.4, using that $\varphi f \in AAP_r^+(\mathbb{R}, X)$ whenever $f \in AAP_r^+(\mathbb{R}, X)$ and $\varphi \in AAP^+(\mathbb{R})$ (Definition 1.2 and [Zai85, p.38])

(ii)⇒(iii): Let $f \in AAP_r^+(\mathbb{R}, X)$ such that $f(s) = 0$ for some $s \in \mathbb{R}$. There exist $g \in AP(\mathbb{R}, X)$ and $h \in C_0^+(\mathbb{R}, X)$ such that $f = g + h$. Then $f = \tilde{g} + \tilde{h}$ where $\tilde{g} := g - \mathbf{1} \otimes g(s)$, $\tilde{h} := h + \mathbf{1} \otimes g(s)$. Clearly, $\tilde{g}$ is an a.p. function, $\tilde{h}(t)$ converges as $t \to \infty$ and $\tilde{g}(s) = \tilde{h}(s) = 0$. From the approximation theorem for a.p. functions [Cor68, Theorem 6.15], we know that there are periodic functions $\tilde{g}_{m,k} : \mathbb{R} \to X$ $(m, k \in \mathbb{N})$ such that

$$\sum_{k=1}^{N(m)} \tilde{g}_{m,k} \to \tilde{g}$$

in $C_b(\mathbb{R}, X)$ as $m \to \infty$, and $\tilde{g}_{m,k}(s) = 0$ for $m, k \in \mathbb{N}$ (otherwise, consider $\tilde{g}_{m,k} - \mathbf{1} \otimes \tilde{g}_{m,k}(s)$). Furthermore, by Lemma 2.3, there exist periodic functions $\varphi_{m,k} \in C_b(\mathbb{R})$, $m, k \in \mathbb{N}$, such that $\varphi_{m,k}(s) = 0$ and

$$\sum_{k=1}^{N(m)} \varphi_{m,k}\tilde{g}_{m,k} \to \tilde{g}$$

in $C_b(\mathbb{R},X)$ as $m\to\infty$. Therefore,

$$T\tilde{g}(s)=\lim_{m\to\infty}\left(T\sum_{k=1}^{N(m)}\varphi_{m,k}\tilde{g}_{m,k}\right)(s)=\lim_{m\to\infty}\sum_{k=1}^{N(m)}\varphi_{m,k}(s)T\tilde{g}_{m,k}(s)=0.$$

It remains to show that $T\tilde{h}(s)=0$. To this end, we choose functions $\phi_n\in C_b(\mathbb{R})$, $n\in\mathbb{N}$, such that $\phi_n(s)=0$ and $\phi_n(r)=1$ for $r\in(-\infty,s-\frac{1}{n}]\cup[s+\frac{1}{n},\infty)$. So, $\phi_n\tilde{h}\to\tilde{h}$ in $C_b(\mathbb{R},X)$ as $n\to\infty$, and we obtain

$$T\tilde{h}(s)=\lim_{n\to\infty}T\phi_n\tilde{h}(s)=\lim_{n\to\infty}\phi_n(s)T\tilde{h}(s)=0.$$

This establishes assertion (ii)⇒(iii). □

Remark 2.6 *The proof shows that Theorem 2.5 remains true if we replace there and in Definition 2.1(ii) the space of a.a.p.r. functions by the space of a.p. or a.a.p. functions.*

3 Evolution semigroups and spectral theory for almost periodic evolution families

In this section, we study spectra for q-p. and a.a.p.r. evolution families, involving evolution semigroup methods. Given a q-p. evolution family $\mathcal{U}$, we consider $V_s:=U(s+q,s)$. It turns out that the spectrum $\sigma(V)$ of the *monodromy operator* $V:=V_0$ coincides with the spectrum of $T(q)$, where $\mathcal{T}$ is the evolution semigroup on the space $P_q(\mathbb{R},X)$. Furthermore, we prove the weak spectral mapping theorem for $\mathcal{T}$. For a.a.p.r. evolution families, we investigate the evolution semigroup on $AAP_r^+(\mathbb{R},X)$ and the spectrum of its generator. In this case, the spectral mapping theorem can be proved as for evolution semigroups on $C_0(\mathbb{R},X)$. The results will be used later in the context of hyperbolic evolution families (Sections 4 and 5).

First, we consider the periodic case and collect the symmetry properties of the spectrum of the evolution semigroup and its generator. From the definition of the evolution semigroup $\mathcal{T}$ for a given q-p. evolution family it is clear that $\mathcal{T}$ is a $\mathcal{C}_0$-semigroup on $\mathcal{F}(\mathbb{R},X)=P_q(\mathbb{R},X)$. Conversely, if the evolution semigroup is given on $P_q(\mathbb{R},X)$, then the corresponding evolution family is q-periodic.

Proposition 3.1 *Let $\mathcal{U}$ be a q-p. evolution family on the Banach space X and let $\mathcal{T}$ be the corresponding evolution semigroup on $P_q(\mathbb{R},X)$ with generator $(G,D(G))$. Then*

(i) $\sigma(T(t))=\left\{e^{\frac{2\pi i}{q}kt}\lambda : k\in\mathbb{Z},\lambda\in\sigma(T(t))\right\}$ *for $t>0$.*

(ii) $\sigma(G)=\sigma(G)+\frac{2\pi i}{q}\mathbb{Z}$.

(iii) $\sigma(T(q))\setminus\{0\}=\sigma(V_s)\setminus\{0\}$ *for all $s\in\mathbb{R}$.*

Proof. Let $\mu_k = \frac{2\pi}{q}k$, $k \in \mathbb{Z}$. Then the operator M_k, given by $M_k f(s) := e^{i\mu_k s} f(s)$ is an isomorphism on $P_q(\mathbb{R}, X)$. Since $M_{-k}T(t)M_k = e^{-i\mu_k t}T(t)$ for $t \geq 0$, we obtain (i) and (ii).

From the q-periodicity of $\mathcal{U}$ and the fact that $\mathcal{T}$ is defined on $P_q(\mathbb{R}, X)$, it follows that $T(q)f(s) = V_s f(s) =: Mf(s)$ for $f \in P_q(\mathbb{R}, X)$. Therefore, $\sigma(T(q))$ coincides with the spectrum of the multiplication operator M. It remains to show that $\sigma(M)\backslash\{0\} = \sigma(V)\backslash\{0\}$.

Let $0 \neq \lambda \in \rho(V)$. Then $\lambda \in \rho(V_s)$ for all $s \in \mathbb{R}$ and $(R(\lambda, V_s))_{s\in\mathbb{R}}$ is a strongly continuous, periodic family of operators and hence uniformly bounded [Rau94, Proposition 12(2)]. So, the multiplication operator $R(\lambda, V.)$ is the bounded inverse of $\lambda - M$.

On the other hand let $0 \neq \lambda \in \rho(M)$. Define $R_\lambda(s)f(s) := (\lambda - M)^{-1}f(s)$ for $s \in \mathbb{R}$ and $f \in P_q(\mathbb{R}, X)$. Let $f \in P_q(\mathbb{R}, X)$ and $s_0 \in \mathbb{R}$ such that $f(s_0) = 0$. Choose continuous functions $\beta_n : [s_0 - \frac{q}{2}, s_0 + \frac{q}{2}) \to [0, 1]$, $0 \leq \beta_n \leq 1$, such that

$$\beta_n(s) = \begin{cases} 0 & : \quad s \in [s_0 - \frac{q}{2}, s_0 - \frac{q}{4n}] \cup [s_0 + \frac{q}{4n}, s_0 + \frac{q}{2}) \\ 1 & : \quad s \in [s_0 - \frac{q}{8n}, s_0 + \frac{q}{8n}] \end{cases}, \quad 0 < n \in \mathbb{N}.$$

Denote by $\alpha_n : \mathbb{R} \to [0, 1]$ the q-p. extension of β_n. Then $\alpha_n R(\lambda, M) = R(\lambda, M)\alpha_n$, $\lim_{n\to\infty} \|\alpha_n f\| = 0$ and $\|R(\lambda, M)f(s_0)\| \leq \|R(\lambda, M)\|\|\alpha_n f\|$. So, $R_\lambda(s_0)f(s_0) = 0$ and therefore $R_\lambda(s)$ is a well-defined bounded, linear operator on X for all $s \in \mathbb{R}$. Since $f(s) = R_\lambda(s)(\lambda - V_s)f(s) = (\lambda - V_s)R_\lambda(s)f(s)$ for all $s \in \mathbb{R}$ and $f \in P_q(\mathbb{R}, X)$, we obtain $R_\lambda(s) = R(\lambda, V_s)$ and therefore $\lambda \in \rho(V_s)$. □

Next, we show that the weak spectral mapping theorem holds for evolution semigroups on $P_q(\mathbb{R}, X)$ (cf. [RS96b, Theorem 2.3] and [LMS96, LMS94] for the spectral mapping theorem on $C_0(\mathbb{R}, X)$). We separate from the proof the following proposition.

Proposition 3.2 *Let $\mathcal{U}$ be a q-p. evolution family on the Banach space X and let $\mathcal{T}$ be the corresponding evolution semigroup on $P_q(\mathbb{R}, X)$ with generator $(G, D(G))$. Then*

$$\sigma(T(q)) \setminus \{0\} = e^{q\sigma(G)}.$$

Proof. It suffices to show that if $1 \in A\sigma(T(q))$, then $0 \in \sigma(G)$ (cf. [RS96b, Section 1]). If $1 \in A\sigma(T(q))$, there exist functions $f_n \in P_q(\mathbb{R}, X)$, $n \in \mathbb{N}$, such that $\|f_n\| = 1$ and $\|T(q)f_n - f_n\| < \frac{1}{n}$. Choose $s_n \in [0, q]$ such that $\|x_n\| := \|f_n(s_n)\| = 1$. Then $\|U(s_n, s_n - q)x_n - x_n\| < \frac{1}{n}$ and hence $\|U(s_n, s_n - q)x_n\| \geq 1 - \frac{1}{n}$. Fix $0 < l < \frac{q}{2}$ and set $I_{s_1} := [s_1, s_1 + q)$. Choose a C^1-function $\beta_1 : I_{s_1} \to [0, 1]$ with bounded derivative such that

$$\beta_1(s) = \begin{cases} 0 & : \quad s \in [s_1, s_1 + l] \\ 1 & : \quad s \in [s_1 + q - l, s_1 + q) \end{cases}.$$

Denote by $\alpha_1 : \mathbb{R} \to [0, 1]$ the periodic extension of β_1. Consider $\alpha_n : \mathbb{R} \to [0, 1]$, $\alpha_n := \alpha_1(s_1 - s_n + \cdot)$ for $n \in \mathbb{N}$. Let $s \in \mathbb{R}$. Then s has a representation $s = s_n + kq + \nu$, where $0 < \nu \leq q$ and $k \in \mathbb{Z}$. Define $g_n : \mathbb{R} \to X$ by means of

$$g_n(s) := (1 - \alpha_n(s))U(s - kq, s_n - q)x_n + \alpha_n(s)U(s - (k + 1)q, s_n - q)x_n.$$

Then $g_n \in P_q(\mathbb{R}, X)$, $n \in \mathbb{N}$. So, for $0 < t < \nu$, we obtain

$$
\begin{aligned}
&\frac{1}{t}(T(t)g_n(s) - g_n(s)) \\
&\quad = \frac{1}{t}[(1-\alpha_n(s-t))U(s_n+\nu, s_n+\nu-t)U(s_n+\nu-t, s_n-q)x_n \\
&\qquad +\alpha_n(s-t)U(s_n+\nu, s_n+\nu-t)U(s_n+\nu-q-t, s_n-q)x_n - g_n(s)] \\
&\quad = \frac{1}{t}[(1-\alpha_n(s-t))U(s_n+\nu, s_n-q)x_n + \alpha_n(s-t)U(s_n+\nu-q, s_n-q)x_n - g_n(s)] \\
&\quad = -\frac{\alpha_n(s-t)-\alpha_n(s)}{t}U(s_n+\nu, s_n)(U(s_n, s_n-q)x_n - x_n) \\
&\quad = -\frac{\beta_1(s_1+\nu-t)-\beta_1(s_1+\nu)}{t}U(s_n+\nu, s_n)(U(s_n, s_n-q)x_n - x_n) \\
&\quad \stackrel{t\searrow 0}{\longrightarrow} -\beta_1'(s_1+\nu)U(s_n+\nu, s_n)(U(s_n, s_n-q)x_n - x_n).
\end{aligned}
$$

This shows that $g_n \in D(G)$ and $\|Gg_n\| \leq \frac{ab}{n}$, where $a := \max\{|\beta_1'(t)| : t \in [s_1, s_1+q]\}$, $b := \max\{\|U(t,s)\| : 0 \leq s \leq t \leq 2q\}$ and $\|g_n\| \geq \|g_n(s_n)\| = \|U(s_n, s_n - q)\| \geq 1 - \frac{1}{n}$. Therefore, $0 \in A\sigma(G) \subset \sigma(G)$. □

Now, we proceed to formulate the desired weak spectral mapping theorem for evolution semigroups on $P_q(\mathbb{R}, X)$.

Theorem 3.3 *Let $\mathcal{U}$ be a q-p. evolution family on the Banach space X and let $\mathcal{T}$ be the corresponding evolution semigroup on $P_q(\mathbb{R}, X)$ with generator $(G, D(G))$. Then*

$$\sigma(T(t)) \setminus \{0\} = \overline{e^{t\sigma(G)}}$$

for all $t \geq 0$.

Proof. We already know from the previous result that $\sigma(T(q)) \setminus \{0\} = e^{q\sigma(G)}$. To conclude the proof, we show that $\sigma(T(t)) \setminus \{0\} = e^{t\sigma(G)}$ for all $0 \leq t \in q\mathbb{Q}$, and pass then to $t \in \mathbb{R}^+$ by taking the closure on the right hand side. So, let $t = \frac{q}{m}$ for some $m \in \mathbb{N}$. From the spectral inclusion theorem and $\sigma(T(q)) \setminus \{0\} = e^{q\sigma(G)}$, we obtain

$$
\begin{aligned}
e^{t\sigma(G)} &\subseteq \sigma(T(t)) \setminus \{0\} \\
&\subseteq \{\sqrt[m]{\mu} : \mu \in \sigma(T(t))^m\} \setminus \{0\} \\
&= \left\{\sqrt[m]{(e^{t\alpha})^m} : \alpha \in \sigma(G)\right\} \\
&= \left\{\sqrt[m]{e^{q\alpha}} : \alpha \in \sigma(G)\right\} \\
&= \left\{e^{\frac{q\,\mathrm{Re}\,\alpha}{m}} e^{i\left(\frac{q\,\mathrm{Im}\,\alpha}{m} + \frac{2\pi}{m}k\right)} : \alpha \in \sigma(G),\ 0 \leq k \leq m-1\right\} \\
&= \left\{e^{t\,\mathrm{Re}\,\alpha} e^{it\,\mathrm{Im}\,\alpha} e^{i\frac{2\pi}{m}k} : \alpha \in \sigma(G),\ 0 \leq k \leq m-1\right\} \\
&= \left\{e^{t\left(\alpha + \frac{2\pi i}{q}k\right)} : \alpha \in \sigma(G),\ 0 \leq k \leq m-1\right\} \\
&= e^{t\sigma(G)},
\end{aligned}
$$

where we used Proposition 3.1(ii) for the last equality. On the other hand, if $\sigma(T(t_0))\setminus\{0\} = e^{t_0\sigma(G)}$ for some $t_0 \in \mathbb{R}^+$, then $\sigma(T(nt_0)) \setminus \{0\} = \sigma(T(t_0))^n \setminus \{0\} = \left(e^{t_0\sigma(G)}\right)^n = e^{nt_0\sigma(G)}$ for $n \in \mathbb{N}$. We therefore obtain $\sigma(T(t)) \setminus \{0\} = e^{t\sigma(G)}$ for $0 \leq t \in q\mathbb{Q}$.

Finally, let $t \in \mathbb{R}^+ \setminus q\mathbb{Q}$ and $r + is = \alpha \in \sigma(G)$. Since $\alpha + \frac{2\pi i}{q}\mathbb{Z} \subseteq \sigma(G)$, it follows from the spectral inclusion theorem that $e^{tr}e^{it\left(s+\frac{2\pi}{q}k\right)} \in \sigma(T(t))$ for all $k \in \mathbb{Z}$. Since $t \notin q\mathbb{Q}$, this implies $e^{tr}\Gamma \subseteq \sigma(T(t))$. On the other hand, if $\sigma(G) \cap i\mathbb{R} = \emptyset$, then we have already seen that $\sigma(T(t)) \cap \Gamma = \emptyset$ for one (hence, for all) $t > 0$. So, from $\sigma(G + \alpha) \cap i\mathbb{R} = \emptyset$, we obtain $\sigma(T(t)) \cap e^{tr}\Gamma = \emptyset$ for all $t > 0$. This shows that $\sigma(T(t))\setminus\{0\} = \overline{e^{t\sigma(G)}}$ for all $t \geq 0$. □

In order to establish corresponding results for evolution semigroups on $AAP_r^+(\mathbb{R}, X)$, we need the following characterization.

Proposition 3.4 *Let $\mathcal{U}$ be an evolution family on the Banach space X. The following assertions are equivalent.*

(i) The evolution family $\mathcal{U}$ is a.a.p.r. and $\lim_{t\to 0} U(s, s-t)x = x$, $x \in X$, uniformly for $s \in \mathbb{R}$.

(ii) The evolution semigroup $\mathcal{T}$ is defined on $AAP_r^+(\mathbb{R}, X)$ and strongly continuous.

Proof. (i)⇒(ii): The semigroup $\mathcal{T}$ acts on $AAP_r^+(\mathbb{R}, X)$. Let $t \in (0, l)$ for fixed $l > 0$. Then

$$\begin{aligned}
&\|T(t)f(s) - f(s)\| \\
&\quad = \|U(s, s-t)(f(s-t) - f(s))\| + \|U(s, s-t)f(s) - f(s)\| \\
&\quad \leq C\|f(s-t) - f(s)\| + \|U(s, s-t)f(s) - f(s)\|
\end{aligned}$$

for $C := \sup\{\|U(s, s-t)\| : t \in (0, l), s \in \mathbb{R}\}$ and $f \in AAP_r^+(\mathbb{R}, X)$. Since the range of an a.a.p.r. function is relatively compact in X and each such function is uniformly continuous, we obtain the strong continuity of $\mathcal{T}$.

(ii)⇒(i): Note that $U(s, s-t)x = T(t)\mathbf{1} \otimes x(s)$. □

Concerning the spectral theory for evolution semigroups on $AAP_r^+(\mathbb{R}, X)$, a main observation is in order. Since $C_0(\mathbb{R}, X)$ is contained in $AAP_r^+(\mathbb{R}, X)$, the spectral mapping theorem and the symmetry properties of the spectra for evolution semigroups on $C_0(\mathbb{R}, X)$ [RS96b] enable us to apply the same arguments for evolution semigroups on $AAP_r^+(\mathbb{R}, X)$. Therefore, we state the following results without proof.

Proposition 3.5 *Let $\mathcal{U}$ be an a.a.p.r. evolution family on the Banach space X such that $\lim_{t\to 0} U(s, s-t)x = x$, $x \in X$, uniformly for $s \in \mathbb{R}$. Let $\mathcal{T}$ be the corresponding evolution semigroup on $AAP_r^+(\mathbb{R}, X)$ with generator $(G, D(G))$. Then*

(i) $\sigma(T(t)) = \Gamma\sigma(T(t))$ for $t > 0$.

(ii) $\sigma(G) = \sigma(G) + i\mathbb{R}$.

Theorem 3.6 *Let $\mathcal{U}$ be an a.a.p.r. evolution family on the Banach space X such that $\lim_{t\to 0} U(s, s-t)x = x$, $x \in X$, uniformly for $s \in \mathbb{R}$. Let $\mathcal{T}$ be the corresponding evolution semigroup on $AAP_r^+(\mathbb{R}, X)$ with generator $(G, D(G))$. Then*

$$\sigma(T(t)) \setminus \{0\} = e^{t\sigma(G)}$$

for all $t \geq 0$.

4 Hyperbolicity of almost periodic evolution families

Hyperbolic evolution families have already been investigated by many authors (see [LMS96, LMS94, LR95, Rau94, RS94, RS96b] and the literature cited therein). Nevertheless, their results did not take into account the particularities one has when dealing with q-p. and a.a.p.r. evolution families. In the following, we characterize hyperbolicity of q-p. evolution families by means of the monodromy operator and the evolution semigroup on $P_q(\mathbb{R}, X)$. In addition, we show a similar result for a.a.p.r. evolution families. The following lemma is the essential step to show that the projection corresponding to a hyperbolic evolution semigroup is a multiplication operator (cf. [Rau94, Lemma 4]).

Lemma 4.1 *(i) Let $\mathcal{T}$ be a hyperbolic evolution semigroup on $P_q(\mathbb{R}, X)$ with projection $\mathcal{P}$. Then $\varphi\mathcal{P}f = \mathcal{P}\varphi f$ for all $\varphi \in P_q(\mathbb{R})$ and $f \in P_q(\mathbb{R}, X)$.*

(ii) Let $\mathcal{T}$ be a hyperbolic evolution semigroup on $AAP_r^+(\mathbb{R}, X)$ with projection $\mathcal{P}$. Then $\varphi\mathcal{P}f = \mathcal{P}\varphi f$ for all $\varphi \in AAP_r^+(\mathbb{R})$ and $f \in AAP_r^+(\mathbb{R}, X)$.

Proof. In order to show (i), consider in the proof of [Rau94, Lemma 4] $P_q(\mathbb{R}, X)$ resp. $P_q(\mathbb{R})$ instead of $C_0(\mathbb{R}, X)$ resp. $C_b(\mathbb{R})$. We mentioned already (see Theorem 2.5) that $\varphi f \in AAP_r^+(\mathbb{R}, X)$ whenever $f \in AAP_r^+(\mathbb{R}, X)$ and $\varphi \in AAP^+(\mathbb{R})$. Reading now again [Rau94, Lemma 4] with $C_0(\mathbb{R}, X)$ and $C_b(\mathbb{R})$ replaced by $AAP_r^+(\mathbb{R}, X)$ and $AAP^+(\mathbb{R})$, respectively, assertion (ii) follows. □

If $\mathcal{U}$ is a periodic evolution family which solves the non-autonomous Cauchy problem (2) for bounded linear operators $A(t)$, then it is known from [DK74, V, Theorem 2.1] that $\sigma(V) \cap \Gamma = \emptyset$ is equivalent to the fact that $\mathcal{U}$ is hyperbolic. We refer to [DK92, Hen81] for partial generalizations of this for unbounded operators $A(t)$. R. Rau showed the above equivalence for general evolution families which are invertible [Rau94].

Theorem 4.2 *Let $\mathcal{U}$ be a q-p. evolution family on the Banach space X and $\mathcal{T}$ be the associated evolution semigroup on $P_q(\mathbb{R}, X)$ with generator G. Then the following assertions are equivalent.*

(i) $\mathcal{U}$ is hyperbolic.

(ii) $\sigma(V) \cap \Gamma = \emptyset$.

(iii) $\mathcal{T}$ *is hyperbolic.*

(iv) $\mathcal{U}$ *is hyperbolic with q-p. projections.*

(v) There exists $\lambda \in i\mathbb{R}$ *such that* $\left[\lambda, \lambda + \frac{2\pi i}{q}\right) \subset \rho(G)$.

Proof. We show (i)⇒(ii)⇒(iii)⇒(iv) and (iii)⇔(v). The implication (iv)⇒(i) is obvious.

(i)⇒(ii): Let $\mathcal{U}$ be hyperbolic. Then the corresponding evolution semigroup $\mathcal{T}_0$ on $C_0(\mathbb{R}, X)$ is hyperbolic [RS94, Theorem 1.5], i.e., $\Gamma \in \rho(T_0(q))$. Since $\Gamma\sigma(V) \setminus \{0\} = \sigma(T_0(q)) \setminus \{0\}$ [Rau94, Proposition 12], we obtain $\sigma(V) \cap \Gamma = \emptyset$.

(ii)⇒(iii): From $\sigma(V) \cap \Gamma = \emptyset$, it follows that $\sigma(T(q)) \cap \Gamma = \emptyset$ (Proposition 3.1 (iii)). Therefore, the evolution semigroup $\mathcal{T}$ is hyperbolic.

(iii)⇒(iv): Let $\mathcal{T}$ be hyperbolic with projection $\mathcal{P}$. Then $\varphi\mathcal{P}f = \mathcal{P}\varphi f$ for all $\varphi \in P_q(\mathbb{R})$ and $f \in P_q(\mathbb{R}, X)$ (Lemma 4.1 (i)). So, from Theorem 2.4, we see that $\mathcal{P}$ is a multiplication operator on $P_q(\mathbb{R}, X)$. Therefore, there are operators $P(t)$, $t \in \mathbb{R}$, such that $\mathcal{P} = P(\cdot) \in P_q(\mathbb{R}, \mathcal{L}_s(X))$. Furthermore, $P(t)$, $t \in \mathbb{R}$, is the desired family of projections for $\mathcal{U}$ (compare the proof of Theorem 1.5(2)⇒(1), case $p = \infty$, in [RS94]).

(iii)⇔(v): This follows from Propositions 3.1 and 3.2. □

For a.a.p.r. evolution families we obtain the following characterization.

Theorem 4.3 *Let* $\mathcal{U}$ *be an a.a.p.r. evolution family on the Banach space* X *such that* $\lim_{t\to 0} U(s, s-t)x = x$, $x \in X$, *uniformly for* $s \in \mathbb{R}$. *Let* $\mathcal{T}$ *be the associated evolution semigroup on* $AAP_r^+(\mathbb{R}, X)$ *with generator* G. *Then the following assertions are equivalent.*

(i) $\mathcal{T}$ *is hyperbolic.*

(ii) $\mathcal{U}$ *is hyperbolic with projections* $P(t)$ *and*

$$R_t f : \mathbb{R} \to X : s \mapsto U_Q(s+t, s)^{-1} Q(s+t) f(s+t)$$

is an a.a.p.r. function for all $f \in AAP_r^+(\mathbb{R}, X)$, $t \geq 0$.

(iii) $0 \in \rho(G)$.

Proof. (i)⇒(ii): Let $\mathcal{T}$ be hyperbolic with projection $\mathcal{P}$. Then $\varphi\mathcal{P}f = \mathcal{P}\varphi f$ for all $\varphi \in AAP_r^+(\mathbb{R})$ and $f \in AAP_r^+(\mathbb{R}, X)$ (Lemma 4.1 (ii)). From Theorem 2.5, we obtain that there are operators $P(t)$, $t \in \mathbb{R}$, such that $\mathcal{P} = P(\cdot) \in AAP_r^+(\mathbb{R}, \mathcal{L}_s(X))$. At this stage, keeping Proposition 3.4 in mind, the arguments of Theorem 1.5, case $p = \infty$, in [RS94] show that $\mathcal{U}$ is hyperbolic with projections $P(t)$. Furthermore, for $\mathcal{Q} = Id - \mathcal{P}$,

$$\mathcal{Q}f(s+t) = \left(T_{\mathcal{Q}}(t)\mathcal{Q}T_{\mathcal{Q}}(t)^{-1}\mathcal{Q}f\right)(s+t) = U_Q(s+t, s)Q(s)\left(T_{\mathcal{Q}}(t)^{-1}\mathcal{Q}f\right)(s)$$

for all $t \geq 0$, $s \in \mathbb{R}$ and $f \in AAP_r^+(\mathbb{R}, X)$. Therefore,

$$R_t f = T_{\mathcal{Q}}(t)^{-1}\mathcal{Q}f \in AAP_r^+(\mathbb{R}, X).$$

(ii)⇒(i): On the other hand, if $\mathcal{U}$ is hyperbolic with projections $P(t)$ such that $R_t f \in AAP_r^+(\mathbb{R}, X)$ for all $f \in AAP_r^+(\mathbb{R}, X)$ and $t \geq 0$, then

$$P(\cdot)x = (Id - R_0)\mathbf{1} \otimes x \in AAP_r^+(\mathbb{R}, X)$$

for all $x \in X$, and $\mathcal{P} = P(\cdot)$ is a bounded projection on $AAP_r^+(\mathbb{R}, X)$ which commutes with $T(t)$, $t \geq 0$. Let $\mathcal{Q} := Id - \mathcal{P}$. Since $T_{\mathcal{P}}(t)\mathcal{P}f = U(\cdot, \cdot - t)P(\cdot - t)f(\cdot - t)$ and $T_{\mathcal{Q}}(t)^{-1}\mathcal{Q}f = R_t f$ whenever $f \in AAP_r^+(\mathbb{R}, X)$, we obtain the hyperbolicity of $\mathcal{T}$, using Definition 1.1(iii) and the boundedness of $P(\cdot)$ and f.

The spectral mapping theorem (Theorem 3.6) and Proposition 3.5 finally show the equivalence of (i) and (iii). □

Remark 4.4 *Let $\mathcal{U}$ be an a.a.p.r. evolution family on the Banach space X such that $\lim_{t\to 0} U(s, s-t)x = x$, $x \in X$, uniformly for $s \in \mathbb{R}$. Let $\mathcal{T}$ be the associated evolution semigroup on $AAP_r^+(\mathbb{R}, X)$ with generator G. It is open whether hyperbolicity of $\mathcal{U}$ is equivalent to the hyperbolicity of $\mathcal{T}$. In particular, we do not know whether the hyperbolicity of $\mathcal{U}$ already implies that $P(\cdot) \in AAP_r^+(\mathbb{R}, \mathcal{L}_s(X))$.*

5 Almost periodicity of inhomogenous Cauchy problems

Let $\mathcal{U}$ be an evolution family and $\mathcal{F}$ one of the spaces $P_q(\mathbb{R}, X)$, $AAP_r^+(\mathbb{R}, X)$. Our aim is the discussion of the following property.

($\mathcal{F}$) *For every $f \in \mathcal{F}$ there exists a unique solution $u \in \mathcal{F}$ of*

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\tau)f(\tau)d\tau, \ t \geq s.$$

For $\mathcal{F} = C_0(\mathbb{R}, X)$, $C_b(\mathbb{R}, X)$, $L^p(\mathbb{R}, X)$, $1 \leq p < \infty$, it is known that property ($\mathcal{F}$) is characterized by the hyperbolicity of $\mathcal{U}$ (see [LRS96] and the literature cited therein). Results in this direction for C_0-semigroups can be found in [Nee96, Prü84]. We show how property ($\mathcal{F}$) is connected to the hyperbolicity of the evolution family $\mathcal{U}$, assuming that $\mathcal{U}$ is almost periodic. It turns out that, in the q-p. case, property ($\mathcal{F}$) can be characterized by $1 \in \rho(V)$. Thereby, $\mathcal{F} = P_q(\mathbb{R}, X)$. In the a.a.p.r. case property ($\mathcal{F}$) for $\mathcal{F} = AAP_r^+(\mathbb{R}, X)$ is equivalent to the hyperbolicity of $\mathcal{U}$ in the sense of Theorem 4.3(ii). First, we need a lemma. For a similar result concerning evolution families on the half-line, we refer to [MRS97, Lemma 1.1].

Lemma 5.1 *Let $\mathcal{T}$ be an evolution semigroup on $\mathcal{F} = P_q(\mathbb{R}, X)$ resp. $\mathcal{F} = AAP_r^+(\mathbb{R}, X)$ with generator $(G, D(G))$. Let $\mathcal{U}$ be the evolution family corresponding to $\mathcal{T}$. Consider $u, f \in \mathcal{F}$. Then the following assertions are equivalent.*

(i) $u \in D(G)$ and $Gu = -f$.

(ii) $u(t) = U(t,s)u(s) + \int_s^t U(t,\tau)f(\tau)d\tau$ *for* $t \geq s$.

Proof. (i)⇒(ii): Let $u \in D(G)$ and $Gu = -f$. Since $\mathcal{T}$ is a $\mathcal{C}_0$-semigroup, we obtain

$$\begin{aligned} -U(t,s)u(s) + u(t) &= -T(t-s)u(t) + u(t) = -\int_0^{t-s} T(\tau)Gu(t)d\tau \\ &= \int_0^{t-s} T(\tau)f(t)d\tau = \int_0^{t-s} U(t,t-\tau)f(t-\tau)d\tau \\ &= \int_s^t U(t,\tau)f(\tau)d\tau \end{aligned}$$

for $t \geq s$.

(ii)⇒(i): Note that

$$\frac{-T(t)u + u}{t} = \frac{1}{t}\int_0^t T(\tau)f d\tau \xrightarrow{t \to 0} f$$

for $u, f \in \mathcal{F}$ satisfying (ii). □

A principal point of Lemma 5.1 is the fact that property $(\mathcal{F})$ can now be characterized by the invertibility of the generator G of the evolution semigroup $\mathcal{T}$ on $\mathcal{F}$.

Corollary 5.2 *Let* $\mathcal{T}$ *be an evolution semigroup on* $\mathcal{F} = P_q(\mathbb{R}, X)$ *resp.* $\mathcal{F} = AAP_r^+(\mathbb{R}, X)$ *with Generator* $(G, D(G))$. *Let* $\mathcal{U}$ *be the evolution family corresponding to* $\mathcal{T}$. *Then the following assertions are equivalent.*

(i) $0 \in \rho(G)$.

(ii) Property $(\mathcal{F})$ *holds.*

Our next result establishes property $(\mathcal{F})$ for $\mathcal{F} = P_q(\mathbb{R}, X)$.

Theorem 5.3 *Let* $\mathcal{U}$ *be a q-p. evolution family on the Banach space* X *and* $\mathcal{F} = P_q(\mathbb{R}, X)$. *Then the following assertions are equivalent.*

(i) Property $(\mathcal{F})$ *holds.*

(ii) $1 \in \rho(V)$.

Proof. Let $\mathcal{T}$ be the corresponding evolution semigroup on $P_q(\mathbb{R}, X)$ with generator G. We have to show that

$$0 \in \rho(G) \Leftrightarrow 1 \in \rho(V)$$

(compare Corollary 5.2). But, this follows from Propositions 3.1 and 3.2. □

Corollary 5.2 also represents the key for connecting hyperbolicity of an a.a.p.r. evolution family with property $(\mathcal{F})$.

Theorem 5.4 *Let $\mathcal{U}$ be an a.a.p.r. evolution family on the Banach space X such that $\lim_{t\to 0} U(s, s-t)x = x$, uniformly for $s \in \mathbb{R}$. Let $\mathcal{F} = AAP_r^+(\mathbb{R}, X)$. The following assertions are equivalent.*

(i) Property ($\mathcal{F}$) holds.

(ii) $\mathcal{U}$ is hyperbolic with projections $P(t)$ and $R_t f \in AAP_r^+(\mathbb{R}, X)$ for all $f \in AAP_r^+(\mathbb{R}, X)$, $t \geq 0$.

Proof. We pass to the evolution semigroup on $AAP_r^+(\mathbb{R}, X)$ with generator G and apply Corollary 5.2 and Theorem 4.3. □

We close this section with two examples.

Example 5.5 *Let Ω be an open bounded set of $\mathbb{R}^n$, with C^2 boundary $\partial\Omega$, and ν the unit vector normal to $\partial\Omega$. Consider*

$$u_t(t,x) - \sum_{i,j=1}^{n} D_i\left(a_{i,j}(t,x)D_j u(t,x)\right) + u(t,x) = f(t,x), \ (t,x) \in \mathbb{R}\times\Omega$$

$$\sum_{i,j=1}^{n} a_{i,j}(t,x)(D_j u(t,x))\nu_i(x) = 0, \ (t,x) \in \mathbb{R}\times\partial\Omega.$$

Suppose that there are $\alpha, \eta > 0$ such that

$$a_{i,j} \in C^{\alpha+1/2}\left(\mathbb{R}, C^0\left(\overline{\Omega}\right)\right) \cap C^{\alpha}\left(\mathbb{R}, C^1\left(\overline{\Omega}\right)\right),$$

$$\operatorname{Re}\left(\sum_{i,j=1}^{n} a_{i,j}(t,x)\overline{\xi_i}\xi\right) \geq \eta|\xi|^2 \ \text{for} \ (t,x) \in \mathbb{R}\times\overline{\Omega}, \ \xi \in \mathbb{C}^n.$$

Suppose further that there exists a real number $q > 0$ such that

$$a_{i,j}(t+q,x) = a_{i,j}(t,x), \ (t,x) \in \mathbb{R}\times\overline{\Omega}.$$

To obtain an abstract Cauchy problem (2), we rewrite this parabolic problem by setting $X := L^2(\Omega)$, $f : \mathbb{R} \to X : t \mapsto f(t,\cdot)$ and

$$D(A(t)) := \left\{ v \in W^{2,2}(\Omega) : \sum_{i,j=1}^{n} a_{i,j}(t,x)(D_j v)\nu_i(x) = 0, \ (t,x) \in \mathbb{R}\times\partial\Omega \right\}$$

$$A(t)v := \sum_{i,j=1}^{n} D_i\left(a_{i,j}(t,x)D_j v\right) + v, \ v \in D(A(t)).$$

It is shown in [Fuh91] that the above assumptions lead to a q-p. evolution family $\mathcal{U}$ induced by the solutions of $u(t) = A(t)u(t)$, $t \in \mathbb{R}$. Let $\mathcal{F} = AAP_r^+(\mathbb{R}, X)$. Then, for every $f \in C_b(\mathbb{R}, X)$ and $\epsilon > 0$ such that $(t \mapsto e^{\epsilon t} f(t)) \in AAP_r^+(\mathbb{R}, X)$, there exists a unique solution u of

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\tau)f(\tau)d\tau, \ t \geq s$$

such that $t \mapsto e^{\epsilon t}u(t)$ is an a.a.p.r. function.

Proof. Consider the q-p. evolution family $\tilde{\mathcal{U}} = \left\{e^{\epsilon(t-s)}U(t,s) : t \geq s\right\}$. Since $\sigma(V) \cap \{\lambda \in \mathbb{C} : |\lambda| = \rho\} = \emptyset$ for all $\rho < 1$ [Fuh91, Section 7], we obtain $\sigma(\tilde{V}) \cap \Gamma = \emptyset$. If we apply Theorems 4.2 and 5.4 for $\tilde{\mathcal{U}}$, the assertion follows. □

Example 5.6 *Let A_0 ba a sectorial operator in X, $0 \leq \alpha < 1$, and $t \mapsto A(t) - A_0 : \mathbb{R} \to \mathcal{L}(X^\alpha, X)$ a locally Hölder continuous a.p. function. Then the corresponding evolution family $\mathcal{U}$ on X fulfils the assumptions of Theorem 5.5 (compare [Hen81, p.240] and [RS96a] for the exponentially boundedness of $\mathcal{U}$). Let $\mathcal{F} = AAP_r^+(\mathbb{R}, X)$. The following assertions are equivalent.*

(i) Property ($\mathcal{F}$) holds.

(ii) $\mathcal{U}$ is hyperbolic.

Proof. Let $\mathcal{U}$ be hyperbolic with projections $P(t)$. Then we obtain $R_t f \in AAP_r^+(\mathbb{R}, X)$ for all $f \in AAP_r^+(\mathbb{R}, X)$, $t \geq 0$ [Hen81, p.240]. □

6 The weakly almost periodic case

The arguments used in Section 2 to Section 5 enable us to replace there $AAP^+(\mathbb{R}, X)$ by the Banach space $W^+(\mathbb{R}, X)$ of weakly almost periodic functions in the sense of Eberlein (E.-w.a.p.). Thereby, a bounded, uniformly continuous functions $f : \mathbb{R} \to X$ is said to be *E.-w.a.p.* if the set of translates $\{(f|_{\mathbb{R}^+})_t : t \in \mathbb{R}^+\}$ is relatively weakly compact in $C_b(\mathbb{R}^+, X)$ [Ebe49, RS89, RV95]. By $W_r^+(\mathbb{R}, X)$ we denote the Banach space of E.-w.a.p. functions with relatively compact range (E.-w.a.p.r.).

We first establish the description of multiplication operators on $W_r^+(\mathbb{R}, X)$ defined analogously by Definition 2.1(ii).

Theorem 6.1 *Let $T \in \mathcal{L}(W_r^+(\mathbb{R}, X))$. Then the following assertions are equivalent.*

(i) The operator T is a multiplication operator on $W_r^+(\mathbb{R}, X)$.

(ii) $T\varphi f = \varphi T f$ for all $f \in W_r^+(\mathbb{R}, X)$ and $\varphi \in W^+(\mathbb{R})$.

(iii) If $f \in W_r^+(\mathbb{R}, X)$ such that $f(s) = 0$ for some $s \in \mathbb{R}$, then $Tf(s) = 0$.

Concerning the discussion of hyperbolic E.-w.a.p.r. evolution families, the spectral mapping theorem for evolution semigroups on $W_r^+(\mathbb{R}, X)$ holds (see Propositions 3.4 and 3.5, Theorem 3.6 and the observation belonging to them).

Theorem 6.2 *Let $\mathcal{U}$ be an E.-w.a.p.r. evolution family on the Banach space X, i.e., the functions $U(t+\cdot, s+\cdot)x$ are E.-w.a.p.r. for all $t \geq s$ and $x \in X$. Let $\mathcal{T}$ be the corresponding evolution semigroup on $W_r^+(\mathbb{R}, X)$ with generator $(G, D(G))$. Then* *(i)* $\sigma(T(t)) = \Gamma\sigma(T(t))$ *for* $t > 0$.

(ii) $\sigma(G) = \sigma(G) + i\mathbb{R}$.

(iii) $\sigma(T(t)) \setminus \{0\} = e^{t\sigma(G)}$ *for all* $t \geq 0$.

We are now well-prepared for the desired hyperbolicity result (cf. Theorem 4.3).

Theorem 6.3 *Let* $\mathcal{U}$ *be an E.-w.a.p.r. evolution family on the Banach space* X *such that* $\lim_{t\to 0} U(s, s-t)x = x$, $x \in X$, *uniformly for* $s \in \mathbb{R}$. *Let* $\mathcal{T}$ *be the associated evolution semigroup on* $W_r^+(\mathbb{R}, X)$ *with generator* G. *Then the following assertions are equivalent.*

(i) $\mathcal{T}$ *is hyperbolic.*

(ii) $\mathcal{U}$ *is hyperbolic with projections* $P(t)$ *and*

$$R_t f : \mathbb{R} \to X : s \mapsto U_Q(s+t, s)^{-1} Q(s+t) f(s+t)$$

is an E.-w.a.p.r. function for all $f \in W_r^+(\mathbb{R}, X)$, $t \geq 0$.

(iii) $0 \in \rho(G)$.

For $\mathcal{F} = W_r^+(\mathbb{R}, X)$, we finally state the connection of property $(\mathcal{F})$ to the hyperbolicity of E.-w.a.p.r. evolution families $\mathcal{U}$.

Theorem 6.4 *Let* $\mathcal{U}$ *be an E.-w.a.p.r. evolution family on the Banach space* X *such that* $\lim_{t\to 0} U(s, s-t)x = x$, *uniformly for* $s \in \mathbb{R}$. *Let* $\mathcal{F} = W_r^+(\mathbb{R}, X)$. *The following assertions are equivalent.*

(i) Property $(\mathcal{F})$ *holds.*

(ii) $\mathcal{U}$ *is hyperbolic with projections* $P(t)$ *and* $R_t f \in W_r^+(\mathbb{R}, X)$ *for all* $f \in W_r^+(\mathbb{R}, X)$, $t \geq 0$.

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