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SPECTRAL MAPPING THEOREMS FOR EVOLUTION SEMIGROUPS ON SPACES OF ALMOST PERIODIC FUNCTIONS

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Abstract. We show that the weak spectral mapping theorem holds for evolution semigroups on spaces of periodic functions. Furthermore, we prove the spectral mapping theorem for evolution semigroups on spaces of (weakly asymptotically) almost periodic functions. The results are applied to the study of hyperbolic periodic and (weakly asymptotically) almost periodic evolution families $\mathcal{U} = \{U(t,s) : t \geq s\}$. We establish conditions on $\mathcal{U}$ which lead to unique periodic resp. (weakly asymptotically) almost periodic mild solutions $u(t) = U(t,s)u(s) + \int_s^t U(t,\tau)f(\tau)d\tau$, $t \geq s \in \mathbb{R}$, provided that f is periodic resp. (weakly asymptotically) almost periodic with relatively compact range.

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1. Introduction and prerequisites. In [Hut97, Remark 2.3.6] the author states that it is not known, whether the spectral mapping theorem holds for evolution semigroups on the space $AP(\mathbb{R}, X)$ of almost periodic functions from $\mathbb{R}$ to a Banach space X. We provide an affirmative answer to this question.

Furthermore, using spectral theory for evolution families and the associated evolution semigroups, we investigate the hyperbolicity of periodic and certain (weakly asymptoticaly) almost periodic evolution families (see [Hut97]). Finally, we apply our results to existence and uniqueness of periodic and almost periodic mild solutions of non-autonomous Cauchy problems.

A set $\mathcal{U} = \{U(t,s) : t \geq s\}$ is called an *evolution family* in the space $\mathcal{L}(X)$ of bounded linear operators on a Banach space X if

(E1) $U(t,t) = Id$, $U(t,r)U(r,s) = U(t,s)$ for $t \geq r \geq s$ in $\mathbb{R}$, (E2) $\{(t,s) \in \mathbb{R}^2 : t \geq s\} \to \mathcal{L}(X) : \ (t,s) \mapsto U(t,s)$ is strongly continuous,

(E3) there are constants $M \geq 1, \omega \in \mathbb{R}$ such that $\|U(t,s)\| \leq Me^{\omega(t-s)}, \ t \geq s$ in $\mathbb{R}$.

For a suitable function $f : \mathbb{R} \to X$, we consider the "variation of constants formula"

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\tau)f(\tau)d\tau, \ t \geq s. \tag{1}$$

In case the evolution family $\mathcal{U}$ is induced by the solutions of a non-autonomous Cauchy problem

$$\dot{u}(t) = A(t)u(t), \ t \in \mathbb{R}, \tag{2}$$

a function u satisfying (1) corresponds to a mild solution of the inhomogeneous equation

$$\dot{u}(t) = A(t)u(t) + f(t), \ t \in \mathbb{R}. \tag{3}$$

Conditions on the operators $A(t)$, $t \in \mathbb{R}$, such that the solutions of (2) define an evolution family $\mathcal{U}$ on X can be found e.g. in [EN00, Fat83, Paz83, Tan79].

We shall be concerned here with evolution families according to the above definition, without assuming the existence of a related Cauchy problem. If $\mathcal{F}(\mathbb{R}, X)$ is a suitable Banach space of functions from $\mathbb{R}$ to X, then, under certain conditions,

$$T(t)f := U(\cdot, \cdot - t)f(\cdot - t), \ t \geq 0, \ f \in \mathcal{F}(\mathbb{R}, X)$$

defines a $\mathcal{C}_0$-semigroup $\mathcal{T} = \{T(t) : t \geq 0\}$ on $\mathcal{F}(\mathbb{R}, X)$, the so called *evolution semigroup* (see [CL99, EN00, How74, Nag95, Rau94a, RRS96, RS96] and the literature cited therein). Let $Q := Id - P$ for a bounded projection P on the Banach space X. Denote by T_Q and T_P the restriction of $T \in \mathcal{L}(X)$ to QX and PX, respectively. Recall that a $\mathcal{C}_0$-semigroup $\mathcal{T}$ on X is called *hyperbolic* if there exists a bounded projection P such that, for $t \geq 0$,

(i) $PT(t) = T(t)P$,

(ii) the operator $T_Q(t)$ is invertible,

(iii) there are constants $N \geq 1$ and $\alpha > 0$ such that

$$\|T_P(t)P\| \leq Ne^{-\alpha t},$$
$$\|T_Q(t)^{-1}Q\| \leq Ne^{-\alpha t}.$$

Hyperbolicity of a $\mathcal{C}_0$-semigroup is characterized by the condition $\Gamma \subseteq \rho(T(t_0))$ for one (hence, for all) $t_0 > 0$, where Γ denotes the unit circle in $\mathbb{C}$ and $\rho(T(t_0))$ is the resolvent set of $T(t_0)$ [Nag86, A-III.3]. Hyperbolicity of an evolution semigroup $\mathcal{T}_0$ on $C_0(\mathbb{R}, X)$ (the Banach space of bounded, continuous functions vanishing at $\pm\infty$) has been considered in [Rau94a, LMS96, LMS94, LR95, RS94, RS96]. For evolution families, we make the following definition (see [LMS96] and [RS96, Section 4]).

DEFINITION 1.1. An evolution family $\mathcal{U}$ is called *hyperbolic* if there are projections $P(t)$, $t \in \mathbb{R}$, such that for $t \geq s$

(i) $P(t)U(t,s) = U(t,s)P(s)$,

(ii) the restriction $U_Q(t,s) : Q(s)X \to Q(t)X$ is invertible,

(iii) there are constants $N \geq 1$ and $\alpha > 0$ such that

$$\begin{aligned} \|U_P(t,s)P(s)\| &\leq Ne^{-\alpha(t-s)}, \\ \|U_Q(t,s)^{-1}Q(t)\| &\leq Ne^{-\alpha(t-s)}. \end{aligned}$$

Note that, if $\mathcal{U}$ is hyperbolic with projections $P(t)$, then $P(\cdot) \in C_b(\mathbb{R}, \mathcal{L}_s(X))$, i.e., $P(\cdot)$ is a bounded, continuous function from $\mathbb{R}$ to $\mathcal{L}(X)$ equipped with the strong operator topology [MRS98, Lemma 4.2]. Our objective is the investigation of hyperbolic evolution families satisfying certain almost periodicity properties. Therefore, we recall some of the notions of almost periodicity which will come into play later on. For a function space $\mathcal{F}(\mathbb{R}, X) \subseteq C_b(\mathbb{R}, X)$, we set

$$\mathcal{F}_r(\mathbb{R}, X) := \{f \in \mathcal{F}(\mathbb{R}, X) : f \text{ has relatively compact range}\}.$$

DEFINITION 1.2. A bounded, continuous function $f \in C_b(\mathbb{R}, X)$ is said to be *q-periodic (q-p.)* $(0 < q \in \mathbb{R})$ if $f(t+q) = f(t)$ for all $t \in \mathbb{R}$. The Banach space of q-p. functions will be denoted by $P_q(\mathbb{R}, X)$.

A bounded, continuous function $f \in C_b(\mathbb{R}, X)$ is said to be *almost periodic* (a.p.) if the set of translates $\{f_\omega := f(\cdot + \omega) : \omega \in \mathbb{R}\}$ is relatively compact in the Banach space $(C_b(\mathbb{R}, X), \|\cdot\|_\infty)$. The Banach space of a.p. functions will be denoted by $AP(\mathbb{R}, X)$ [Boh32, Boc33, Cor68].

A function $f : \mathbb{R} \to X$ is said to be *asymptotically almost periodic* (a.a.p.) if there is an a.p. function g and a function

$$h \in C_0^+(\mathbb{R}, X) := \{f : \mathbb{R} \to X : f \text{ is bounded, uniformly continuous, and } \lim_{t\to\infty} \|f(t)\| = 0\}$$

such that $f = g + h$. The Banach space of a.a.p. functions will be denoted by $AAP^+(\mathbb{R}, X)$ [Fré41, RS88, Zai85, RV95]. We also write *a.a.p.r. function* for a function belonging to the Banach space $AAP_r^+(\mathbb{R}, X)$.

A function $f \in C_b(\mathbb{R}, X)$ is said to be *weakly almost periodic in the sense of Eberlein* (E.-w.a.p.) if the set of translates $\{f_t : t \in \mathbb{R}\}$ is relatively weakly compact in $C_b(\mathbb{R}, X)$. The Banach space of E.-w.a.p. functions will be denoted by $W(\mathbb{R}, X)$. We also write E.-w.a.p.r. function for a function belonging to the Banach space $W_r(\mathbb{R}, X)$.

A bounded, uniformly continuous function $f \in BUC(\mathbb{R}, X)$ is said to be *weakly asymptotically almost periodic in the sense of Eberlein* (E.-w.a.a.p.) if the set of translates $\{(f|_{\mathbb{R}^+})_t : t \in \mathbb{R}^+\}$ is relatively weakly compact in $BUC(\mathbb{R}^+, X)$. The Banach space of E.-w.a.a.p. functions will be denoted by $W^+(\mathbb{R}, X)$. We also write E.-w.a.a.p.r. function for a function belonging to the Banach space $W_r^+(\mathbb{R}, X)$.

We take for short '**ap**' to denote one of the properties a.p., a.a.p.r., E.-w.a.p.r. and E.-w.a.a.p.r., respectively. Correspondingly, we denote by $\mathcal{A}(\mathbb{R}, X)$ one of the spaces $AP(\mathbb{R}, X)$, $AAP_r^+(\mathbb{R}, X)$, $W_r(\mathbb{R}, X)$ and $W_r^+(\mathbb{R}, X)$, respectively.

An operator-valued function $P(\cdot) : \mathbb{R} \to \mathcal{L}(X)$ is called q-p. resp. **ap** if $P(\cdot)x$ is q-p. resp. **ap** for all $x \in X$. The Banach space of q-p. resp. **ap** operator-valued functions will be denoted by $P_q(\mathbb{R}, \mathcal{L}_s(X))$ resp. $\mathcal{A}(\mathbb{R}, \mathcal{L}_s(X))$.

In a preliminary section (Section 2), we study multiplication operators on spaces of almost periodic functions. Then, in Section 3, our interest is directed to q-p. resp. **ap** evolution families $\mathcal{U}$, i.e.,

$$U(t + \cdot, s + \cdot)x : \mathbb{R} \to X$$

is a q-p. resp. **ap** function for all $t \geq s$ and $x \in X$. We introduce evolution semigroups corresponding to such evolution families and prove the spectral mapping theorems which will be used in our discussion of hyperbolicity in the fourth section (see [Hut97]). We obtain characterizations of hyperbolic q-p. evolution families and corresponding results for **ap** evolution families. In Section 5, we turn to existence and uniqueness of q-p. resp. **ap** solutions of the integral equation (1).

Let us fix some notations. For a linear operator A, denote by $A\sigma(A)$ resp. $\sigma(A)$ the approximate point spectrum resp. the spectrum of A. For the constant function $x \in X$ we also write $\mathbf{1} \otimes x$.

2. Multiplication operators on spaces of almost periodic functions. In this section, we are concerned with multiplication operators on the space of q-p. resp. **ap** functions (cf. [Gra97, Rau94a] for multiplication operators on $C_0(\mathbb{R}, X)$). We give characterizations which will be used in the next section when dealing with evolution semigroups defined on these spaces.

DEFINITION 2.1. Let $\mathcal{F}(\mathbb{R}, X)$ be one of the spaces $P_q(\mathbb{R}, X)$ resp. $\mathcal{A}(\mathbb{R}, X)$, An operator $T \in \mathcal{L}(\mathcal{F}(\mathbb{R}, X))$ is called a *multiplication operator on* $\mathcal{F}(\mathbb{R}, X)$ if there exists a function $F : \mathbb{R} \to \mathcal{L}(X)$ such that $(Tf)(s) = F(s)f(s)$ for all $f \in \mathcal{F}(\mathbb{R}, X)$ and $s \in \mathbb{R}$.

In order to characterize such operators, the following observation turns out to be useful.

LEMMA 2.2. *Let $\mathcal{F}$ stand for P_q resp. $\mathcal{A}$. Let $T \in \mathcal{L}(\mathcal{F}(\mathbb{R}, X))$ and $F : \mathbb{R} \to \mathcal{L}(X)$ such that $(Tf)(s) = F(s)f(s)$ for all $s \in \mathbb{R}$ and $f \in \mathcal{F}(\mathbb{R}, X)$. Then $F \in \mathcal{F}(\mathbb{R}, \mathcal{L}_s(X))$.*

Proof. Note that $F(s)x = (T\mathbf{1} \otimes x)(s)$ for $x \in X$ and $s \in \mathbb{R}$. □

Furthermore, we need an approximation result for periodic functions.

LEMMA 2.3. *Let $f \in P_q(\mathbb{R}, X)$ and suppose that $f(s) = 0$ for some $s \in \mathbb{R}$. Then there exist functions $\varphi_n \in P_q(\mathbb{R})$, $n \in \mathbb{N}$, such that $\varphi_n(s) = 0$ and $\varphi_n f \to f$ in $C_b(\mathbb{R}, X)$ as $n \to \infty$.*

Proof. Let $s \in \mathbb{R}$ such that $f(s) = 0$ and choose functions $\varphi_n \in P_q(\mathbb{R}, X)$ such that

$$\varphi_n(t) = \begin{cases} 1 & : \quad t \in [s + \frac{1}{n}, s + q - \frac{1}{n}] \\ 0 & : \quad t = s \end{cases}, \quad n \in \mathbb{N}.$$

This immediately yields the assertion. □

Keeping the above considerations in mind, we obtain the following description of multiplication operators on $P_q(\mathbb{R}, X)$.

THEOREM 2.4. *Let $T \in \mathcal{L}(P_q(\mathbb{R}, X))$. Then the following assertions are equivalent.*

(i) The operator T is a multiplication operator on $P_q(\mathbb{R}, X)$.

(ii) $T\varphi f = \varphi T f$ for all $f \in P_q(\mathbb{R}, X)$ and $\varphi \in P_q(\mathbb{R})$.

(iii) If $f \in P_q(\mathbb{R}, X)$ such that $f(s) = 0$ for some $s \in \mathbb{R}$, then $(Tf)(s) = 0$.

Proof. (i)⇒(ii): Let $F \in P_q(\mathbb{R}, \mathcal{L}_s(X))$ be the function representing the operator T. Then, for $f \in P_q(\mathbb{R}, X)$ and $\varphi \in P_q(\mathbb{R})$, it follows that

$$(T\varphi f)(s) = F(s)\varphi(s)f(s) = \varphi(s)F(s)f(s) = \varphi(s)(Tf)(s)$$

for all $s \in \mathbb{R}$.

(ii)⇒(iii): Let $f \in P_q(\mathbb{R}, X)$ such that $f(s) = 0$ for some $s \in \mathbb{R}$. Let φ_n, $n \in \mathbb{N}$, be given as in Lemma 2.3. Then

$$(Tf)(s) = \lim_{n\to\infty} (T\varphi_n f)(s) = \lim_{n\to\infty} \varphi_n(s)(Tf)(s) = 0.$$

(iii)⇒(i): Define $F : \mathbb{R} \to \mathcal{L}(X)$ by means of $F(s)x := (Tf)(s)$ where $f \in P_q(\mathbb{R}, X)$ satisfies $f(s) = x$. Note that (iii) implies that $F(s)x$ is well-defined. □

The a.a.p.r. version of this result can be obtained by similar arguments, using the approximation theorem for a.p. functions.

THEOREM 2.5. *Let $T \in \mathcal{L}(AAP_r^+(\mathbb{R}, X))$. Then the following assertions are equivalent.*

(i) The operator T is a multiplication operator on $AAP_r^+(\mathbb{R}, X)$.

(ii) $T\varphi f = \varphi T f$ for all $f \in AAP_r^+(\mathbb{R}, X)$ and $\varphi \in AAP^+(\mathbb{R})$.

(iii) If $f \in AAP_r^+(\mathbb{R}, X)$ such that $f(s) = 0$ for some $s \in \mathbb{R}$, then $Tf(s) = 0$.

Proof. The implications (i)⇒(ii) and (iii)⇒(i) can be shown as the corresponding implications in Theorem 2.4, using that $\varphi f \in AAP_r^+(\mathbb{R}, X)$ whenever $f \in AAP_r^+(\mathbb{R}, X)$ and $\varphi \in AAP^+(\mathbb{R})$ [Zai85, p. 38].

(ii)⇒(iii): Let $f \in AAP_r^+(\mathbb{R}, X)$ such that $f(s) = 0$ for some $s \in \mathbb{R}$. There exist $g \in AP(\mathbb{R}, X)$ and $h \in C_0^+(\mathbb{R}, X)$ such that $f = g + h$. Then $f = \tilde{g} + \tilde{h}$

where $\tilde{g} := g - \mathbf{1} \otimes g(s)$, $\tilde{h} := h + \mathbf{1} \otimes g(s)$. Clearly, $\tilde{g}$ is an a.p. function, $\tilde{h}(t)$ is convergent as $t \to \infty$ and $\tilde{g}(s) = \tilde{h}(s) = 0$. From the approximation theorem for a.p. functions [Cor68, Theorem 6.15], we know that there are periodic functions $\tilde{g}_{m,k} : \mathbb{R} \to X$ $(m, k \in \mathbb{N})$ such that

$$\sum_{k=1}^{N(m)} \tilde{g}_{m,k} \to \tilde{g}$$

in $C_b(\mathbb{R}, X)$ as $m \to \infty$, and $\tilde{g}_{m,k}(s) = 0$ for $m, k \in \mathbb{N}$ (otherwise, consider $\tilde{g}_{m,k} - \mathbf{1} \otimes \tilde{g}_{m,k}(s)$). Furthermore, by Lemma 2.3, there exist periodic functions $\varphi_{m,k} \in C_b(\mathbb{R})$, $m, k \in \mathbb{N}$, such that $\varphi_{m,k}(s) = 0$ and

$$\sum_{k=1}^{N(m)} \varphi_{m,k} \tilde{g}_{m,k} \to \tilde{g}$$

in $C_b(\mathbb{R}, X)$ as $m \to \infty$. Therefore,

$$\begin{aligned} (T\tilde{g})(s) &= \lim_{m\to\infty} \left(T \sum_{k=1}^{N(m)} \varphi_{m,k} \tilde{g}_{m,k} \right)(s) \\ &= \lim_{m\to\infty} \sum_{k=1}^{N(m)} \varphi_{m,k}(s) \left(T\tilde{g}_{m,k}\right)(s) = 0. \end{aligned}$$

It remains to show that $(T\tilde{h})(s) = 0$. To this end, we choose functions $\phi_n \in C_b(\mathbb{R})$, $n \in \mathbb{N}$, such that $\phi_n(s) = 0$ and $\phi_n(r) = 1$ for $r \in (-\infty, s - \frac{1}{n}] \cup [s + \frac{1}{n}, \infty)$. So, $\phi_n \tilde{h} \to \tilde{h}$ in $C_b(\mathbb{R}, X)$ as $n \to \infty$, and we obtain

$$\begin{aligned} (T\tilde{h})(s) &= \lim_{n\to\infty} \left(T\phi_n \tilde{h} \right)(s) \\ &= \lim_{n\to\infty} \phi_n(s)(T\tilde{h})(s) = 0. \end{aligned}$$

This establishes assertion (ii)⇒(iii). □

REMARK 2.6. The proof shows that Theorem 2.5 remains true if we replace the space of a.a.p.r. functions by the space of a.p. functions.

For multiplication operators on the space of E.-w.a.p.r. resp. E.-w.a.a.p.r. functions, we obtain an analogous characterization.

THEOREM 2.7. *Let $T \in \mathcal{L}(W_r(\mathbb{R}, X))$ resp. $T \in \mathcal{L}(W_r^+(\mathbb{R}, X))$. Then the following assertions are equivalent.*

(i) The operator T is a multiplication operator on $W_r(\mathbb{R}, X)$ resp. $W_r^+(\mathbb{R}, X)$.

(ii) $T\varphi f = \varphi T f$ for all $f \in W_r(\mathbb{R}, X)$ and $\varphi \in W(\mathbb{R})$ resp. $f \in W_r^+(\mathbb{R}, X)$ and $\varphi \in W^+(\mathbb{R})$.

(iii) *If $f \in W_r(\mathbb{R},X)$ resp. $f \in W_r^+(\mathbb{R},X)$ such that $f(s)=0$ for some $s \in \mathbb{R}$, then $Tf(s)=0$.*

Proof. This can be shown as in Theorem 2.5, using that $\varphi f \in W_r^{(+)}(\mathbb{R},X)$ whenever $f \in W_r^{(+)}(\mathbb{R},X)$ and $\varphi \in W^{(+)}(\mathbb{R})$ [Kr92, Corollary 2.5]. □

3. Evolution semigroups and spectral theory for evolution families satisfying almost periodicity properties. In this section, we study the spectra for q-p. and **ap** evolution families, involving evolution semigroup methods. Given a q-p. evolution family $\mathcal{U}$, we define $V_s := U(s+q,s)$. It turns out that the spectrum $\sigma(V)$ of the *monodromy operator* $V := V_0$ coincides with the spectrum of $T(q)$, where $\mathcal{T}$ is the evolution semigroup on the space $P_q(\mathbb{R},X)$. Furthermore, we prove the weak spectral mapping theorem for $\mathcal{T}$. For **ap** evolution families, we investigate the evolution semigroup on $\mathcal{A}(\mathbb{R},X)$ and the spectrum of its generator. We show that the spectral mapping theorem holds for evolution semigroups on $AP(\mathbb{R},X)$. For evolution semigroups on $AAP^+(\mathbb{R},X)$, $W_r(\mathbb{R},X)$ resp. $W_r^+(\mathbb{R},X)$, the spectral mapping theorem can be proved as for evolution semigroups on $C_0(\mathbb{R},X)$.

First, we are concerned with periodic evolution families. We collect the symmetry properties of the spectrum of the corresponding evolution semigroup and its generator (see [Rau94b, Proposition 6] and [LMS96, Theorem 3.1] for evolution semigroups considered on $C_0(\mathbb{R},X)$ and $L^p(\mathbb{R},X)$). Note that from the definition of the evolution semigroup $\mathcal{T}$ for a given q-p. evolution family it is clear that $\mathcal{T}$ is a $\mathcal{C}_0$-semigroup on $\mathcal{F}(\mathbb{R},X)=P_q(\mathbb{R},X)$. Conversely, if the evolution semigroup is given on $P_q(\mathbb{R},X)$, then the corresponding evolution family is q-periodic.

PROPOSITION 3.1. *Let $\mathcal{U}=\{U(t,s): t \geq s \in \mathbb{R}\}$ be a q-p. evolution family on the Banach space X and let $\mathcal{T}$ be the corresponding evolution semigroup on $P_q(\mathbb{R},X)$ with generator $(G,D(G))$. Then*

(i) $\sigma(T(t)) = \left\{ e^{\frac{2\pi i}{q}kt}\lambda : k \in \mathbb{Z}, \lambda \in \sigma(T(t)) \right\}$ *for* $t>0$.

(ii) $\sigma(G) = \sigma(G) + \frac{2\pi i}{q}\mathbb{Z}$.

(iii) $\sigma(T(q)) \setminus \{0\} = \sigma(V_s) \setminus \{0\}$ *for all* $s \in \mathbb{R}$.

Proof. Let $\mu_k = \frac{2\pi}{q}k$, $k \in \mathbb{Z}$. Then the operator M_k, given by $(M_k f)(s) := e^{i\mu_k s} f(s)$ is an isomorphism on $P_q(\mathbb{R},X)$. Since $M_{-k}T(t)M_k = e^{-i\mu_k t}T(t)$ for $k \in \mathbb{Z}$, we obtain (i) and (ii).

Recall that $\sigma(V_s)\setminus\{0\}$ is independent of $s \in \mathbb{R}$ [DK92, Proposition 6.3]. From the q-periodicity of $\mathcal{U}$ and the fact that $\mathcal{T}$ is defined on $P_q(\mathbb{R},X)$, it follows that $(T(q)f)(s) = V_s f(s) =: (Mf)(s)$ for $f \in P_q(\mathbb{R},X)$. Therefore, $\sigma(T(q))$ coincides with the spectrum of the multiplication operator M. It remains to show that $\sigma(M)\setminus\{0\} = \sigma(V)\setminus\{0\}$.

Let $0 \neq \lambda \in \rho(V)$. Then $\lambda \in \rho(V_s)$ for all $s \in \mathbb{R}$, and $(R(\lambda,V_s))_{s\in\mathbb{R}}$ is a strongly continuous, periodic family of operators and hence uniformly bounded [Rau94a, Proposition 12(2)]. So, the multiplication operator $R(\lambda,V_\cdot)$ is the bounded inverse of $\lambda - M$.

On the other hand, let $0 \neq \lambda \in \rho(M)$. Define $R_\lambda(s)f(s) := \left((\lambda - M)^{-1}f\right)(s)$ for $s \in \mathbb{R}$ and $f \in P_q(\mathbb{R}, X)$. Let $f \in P_q(\mathbb{R}, X)$ and $s_0 \in \mathbb{R}$ such that $f(s_0) = 0$. Choose continuous functions $\beta_n : [s_0 - \frac{q}{2}, s_0 + \frac{q}{2}) \to [0,1]$, $0 \leq \beta_n \leq 1$, such that

$$\beta_n(s) = \begin{cases} 0 & : \quad s \in [s_0 - \frac{q}{2}, s_0 - \frac{q}{4n}] \cup [s_0 + \frac{q}{4n}, s_0 + \frac{q}{2}) \\ 1 & : \quad s \in [s_0 - \frac{q}{8n}, s_0 + \frac{q}{8n}] \end{cases}, \quad 0 < n \in \mathbb{N}.$$

Denote by $\alpha_n : \mathbb{R} \to [0,1]$ the q-p. extension of β_n. Then $\alpha_n R(\lambda, M) = R(\lambda, M)\alpha_n$, $\lim_{n\to\infty} \|\alpha_n f\| = 0$ and $\|(R(\lambda, M)f)(s_0)\| \leq \|R(\lambda, M)\| \|\alpha_n f\|$. So, $R_\lambda(s_0)f(s_0) = 0$ and therefore $R_\lambda(s)$ is a well-defined bounded, linear operator on X for all $s \in \mathbb{R}$. Since $f(s) = R_\lambda(s)(\lambda - V_s)f(s) = (\lambda - V_s)R_\lambda(s)f(s)$ for all $s \in \mathbb{R}$ and $f \in P_q(\mathbb{R}, X)$, we obtain $R_\lambda(s) = R(\lambda, V_s)$ and therefore $\lambda \in \rho(V_s)$. □

Next, we show that the weak spectral mapping theorem holds for evolution semigroups on $P_q(\mathbb{R}, X)$ (cf. [RS96, Theorem 2.3] and [LMS94, LMS96] for the spectral mapping theorem on $C_0(\mathbb{R}, X)$). We separate from the proof the following proposition which is of independent interest.

PROPOSITION 3.2. *Let $\mathcal{U} = \{U(t,s) : t \geq s \in \mathbb{R}\}$ be a q-p. evolution family on the Banach space X, and let $\mathcal{T}$ be the corresponding evolution semigroup on $P_q(\mathbb{R}, X)$ with generator $(G, D(G))$. Then*

$$\sigma(T(q)) \setminus \{0\} = e^{q\sigma(G)}.$$

Proof. Let $A\sigma(T(q))$ be the approximate point spectrum of $T(q)$. It suffices to show that if $1 \in A\sigma(T(q))$, then $0 \in \sigma(G)$ (cf. [RS96, Section 1]). If $1 \in A\sigma(T(q))$, there exist functions $f_n \in P_q(\mathbb{R}, X)$, $n \in \mathbb{N}$, such that $\|f_n\| = 1$ and $\|T(q)f_n - f_n\| < \frac{1}{n}$. Choose $s_n \in [0, q]$ such that $\|x_n\| := \|f_n(s_n)\| = 1$. Then $\|U(s_n, s_n - q)x_n - x_n\| < \frac{1}{n}$ and hence $\|U(s_n, s_n - q)x_n\| \geq 1 - \frac{1}{n}$. Fix $0 < l < \frac{q}{2}$ and set $I_{s_1} := [s_1, s_1 + q)$. Choose a C^1-function $\beta_1 : I_{s_1} \to [0,1]$ with bounded derivative such that

$$\beta_1(s) = \begin{cases} 0 & : \quad s \in [s_1, s_1 + l] \\ 1 & : \quad s \in [s_1 + q - l, s_1 + q) \end{cases}.$$

Denote by $\alpha_1 : \mathbb{R} \to [0,1]$ the periodic extension of β_1. Consider $\alpha_n : \mathbb{R} \to [0,1]$, $\alpha_n := \alpha_1(s_1 - s_n + \cdot)$ for $n \in \mathbb{N}$. Let $s \in \mathbb{R}$. Then s has a representation $s = s_n + kq + \nu$, where $0 < \nu \leq q$ and $k \in \mathbb{Z}$. Define $g_n : \mathbb{R} \to X$ by means of

$$g_n(s) := (1 - \alpha_n(s))U(s - kq, s_n - q)x_n + \alpha_n(s)U(s - (k+1)q, s_n - q)x_n.$$

Then $g_n \in P_q(\mathbb{R}, X)$, $n \in \mathbb{N}$. So, for $0 < t < \nu$, we obtain

$$\begin{aligned}
&\frac{1}{t}((T(t)g_n)(s) - g_n(s)) \\
&\quad = \frac{1}{t}[(1-\alpha_n(s-t))U(s_n+\nu, s_n+\nu-t)U(s_n+\nu-t, s_n-q)x_n \\
&\qquad + \alpha_n(s-t)U(s_n+\nu, s_n+\nu-t)U(s_n+\nu-q-t, s_n-q)x_n \\
&\qquad - g_n(s)] \\
&\quad = \frac{1}{t}[(1-\alpha_n(s-t))U(s_n+\nu, s_n-q)x_n \\
&\qquad + \alpha_n(s-t)U(s_n+\nu-q, s_n-q)x_n - g_n(s)] \\
&\quad = -\frac{\alpha_n(s-t)-\alpha_n(s)}{t}U(s_n+\nu, s_n)(U(s_n, s_n-q)x_n - x_n) \\
&\quad = -\frac{\beta_1(s_1+\nu-t)-\beta_1(s_1+\nu)}{t}U(s_n+\nu, s_n)(U(s_n, s_n-q)x_n - x_n) \\
&\quad \xrightarrow{t\searrow 0} -\beta_1'(s_1+\nu)U(s_n+\nu, s_n)(U(s_n, s_n-q)x_n - x_n)
\end{aligned}$$

uniformly for $s \in \mathbb{R}$. This shows that $g_n \in D(G)$ and $\|Gg_n\| \le \frac{ab}{n}$, where $a := \max\{|\beta_1'(t)| : t \in [s_1, s_1+q]\}$, $b := \max\{\|U(t,s)\| : 0 \le s \le t \le 2q\}$ and $\|g_n\| \ge \|g_n(s_n)\| = \|U(s_n, s_n-q)\| \ge 1 - \frac{1}{n}$. Therefore, $0 \in A\sigma(G) \subset \sigma(G)$. □

We now show the announced weak spectral mapping theorem for evolution semigroups on $P_q(\mathbb{R}, X)$.

Theorem 3.3. *Let $\mathcal{U} = \{U(t,s) : t \ge s \in \mathbb{R}\}$ be a q-p. evolution family on the Banach space X, and let $\mathcal{T}$ be the corresponding evolution semigroup on $P_q(\mathbb{R}, X)$ with generator $(G, D(G))$. Then*

$$\sigma(T(t)) \setminus \{0\} = \overline{e^{t\sigma(G)}}$$

for all $t \ge 0$.

Proof. We already know from the previous result that $\sigma(T(q)) \setminus \{0\} = e^{q\sigma(G)}$. Next, we show that

$$\sigma(T(t)) \setminus \{0\} = e^{t\sigma(G)} \text{ for all } 0 \le t \in q\mathbb{Q}.$$

At first, let $t = \frac{q}{m}$ for some $m \in \mathbb{N}$. From the spectral inclusion theorem [Nag86,

p. 84] and $\sigma(T(q)) \setminus \{0\} = e^{q\sigma(G)}$, we obtain

$$\begin{aligned}
e^{t\sigma(G)} &\subseteq \sigma(T(t)) \setminus \{0\} \\
&\subseteq \{ \sqrt[m]{\mu} : \mu \in \sigma(T(t))^m \} \setminus \{0\} \\
&= \left\{ \sqrt[m]{(e^{t\alpha})^m} : \alpha \in \sigma(G) \right\} \\
&= \left\{ \sqrt[m]{e^{q\alpha}} : \alpha \in \sigma(G) \right\} \\
&= \left\{ e^{\frac{q\,\mathrm{Re}\alpha}{m}} e^{i\left(\frac{q\,\mathrm{Im}\alpha}{m} + \frac{2\pi}{m}k\right)} : \alpha \in \sigma(G),\ 0 \leq k \leq m-1 \right\} \\
&= \left\{ e^{t\,\mathrm{Re}\,\alpha} e^{it\,\mathrm{Im}\,\alpha} e^{i\frac{2\pi}{m}k} : \alpha \in \sigma(G),\ 0 \leq k \leq m-1 \right\} \\
&= \left\{ e^{t\left(\alpha + \frac{2\pi i}{q}k\right)} : \alpha \in \sigma(G),\ 0 \leq k \leq m-1 \right\} \\
&= e^{t\sigma(G)},
\end{aligned}$$

where we used Proposition 3.1(ii) for the last equality. On the other hand, if $\sigma(T(t_0)) \setminus \{0\} = e^{t_0\sigma(G)}$ for some $t_0 \in \mathbb{R}^+$, then $\sigma(T(nt_0)) \setminus \{0\} = \sigma(T(t_0))^n \setminus \{0\} = \left(e^{t_0\sigma(G)}\right)^n = e^{nt_0\sigma(G)}$ for $n \in \mathbb{N}$. We therefore obtain $\sigma(T(t)) \setminus \{0\} = e^{t\sigma(G)}$ for all $0 \leq t \in q\mathbb{Q}$.

Finally, let $t \in \mathbb{R}^+ \setminus q\mathbb{Q}$ and let $r + is = \alpha \in \sigma(G)$ with $r, s \in \mathbb{R}$. Since $\alpha + \frac{2\pi i}{q}\mathbb{Z} \subseteq \sigma(G)$, it follows from the spectral inclusion theorem that $e^{tr} e^{it\left(s + \frac{2\pi}{q}k\right)} \in \sigma(T(t))$ for all $k \in \mathbb{Z}$. Since $t \notin q\mathbb{Q}$, this implies $e^{tr}\Gamma \subseteq \sigma(T(t))$. On the other hand, if $\sigma(G) \cap i\mathbb{R} = \emptyset$, then we have already seen that $\sigma(T(q)) \cap \Gamma = \emptyset$. Hence, $\sigma(T(t)) \cap \Gamma = \emptyset$ for all $t > 0$ (see [Nag86, A-III.3]). So, from $\sigma(G + \alpha) \cap i\mathbb{R} = \emptyset$, $\alpha = r + is$, $r, s \in \mathbb{R}$, we obtain $\sigma(T(t)) \cap e^{tr}\Gamma = \emptyset$ for all $t > 0$. This shows that $\sigma(T(t)) \setminus \{0\} = \overline{e^{t\sigma(G)}}$ for all $t \geq 0$. □

Throughout the remainder of this section, we discuss related spectral properties of evolution semigroups on spaces of almost periodic functions. Note that evolution semigroups on the spaces $AP(\mathbb{R}, X)$ and $W_r(\mathbb{R}, X)$ were considered in [BGP89] and [Kr92], respectively. For our situation, we need the following characterization.

Proposition 3.4. *Let $\mathcal{U} = \{U(t,s) : t \geq s \in \mathbb{R}\}$ be an evolution family on the Banach space X. The following assertions are equivalent.*

(i) The evolution family $\mathcal{U}$ is **ap** *and*

$$\lim_{t \to 0} U(s, s-t)x = x \text{ for all } x \in X,$$

uniformly for $s \in \mathbb{R}$.

(ii) The semigroup $\mathcal{T}$ is defined on $\mathcal{A}(\mathbb{R}, X)$ and strongly continuous, i.e., $\mathcal{T}$ is an evolution semigroup on $\mathcal{A}(\mathbb{R}, X)$.

Proof. (i)⇒(ii): The semigroup $\mathcal{T}$ acts on $\mathcal{A}(\mathbb{R}, X)$ whenever $U(t + \cdot, s + \cdot)x \in \mathcal{A}(\mathbb{R}, X)$ for all $t \geq s \in \mathbb{R}$ and $x \in X$. This follows from [Kr92, Corollary 2.5] and

[Zai85, p. 30, Example 1]. Let $t \in (0, l)$ for fixed $l > 0$. Then

$$\begin{aligned}&\|(T(t)f)(s) - f(s)\| \\ &\quad = \|U(s, s-t)(f(s-t) - f(s))\| + \|U(s, s-t)f(s) - f(s)\| \\ &\quad \leq C\|f(s-t) - f(s)\| + \|U(s, s-t)f(s) - f(s)\|\end{aligned}$$

for $C := \sup\{\|U(s, s-t)\| : t \in (0, l), s \in \mathbb{R}\}$ and $f \in \mathcal{A}(\mathbb{R}, X)$. Since the range of a function belonging to $\mathcal{A}(\mathbb{R}, X)$ is relatively compact in X and each such function is uniformly continuous, we obtain the strong continuity of $\mathcal{T}$.

(ii)⇒(i): Note that $U(s, s-t)x = (T(t)\mathbf{1} \otimes x)(s)$. □

We formulate the following proposition (see also [Rau94b, Proposition 6] and [LMS96, Theorem 3.1]).

PROPOSITION 3.5. *Let $\mathcal{U} = \{U(t, s) : t \geq s \in \mathbb{R}\}$ be an* **ap** *evolution family on the Banach space X such that $\lim_{t\to 0} U(s, s-t)x = x$, $x \in X$, uniformly for $s \in \mathbb{R}$. Let $\mathcal{T}$ be the corresponding evolution semigroup on $\mathcal{A}(\mathbb{R}, X)$ with generator $(G, D(G))$. Then*

(i) $\sigma(T(t)) = \Gamma\sigma(T(t))$ for $t > 0$.

(ii) $\sigma(G) = \sigma(G) + i\mathbb{R}$.

Proof. The operator M_μ, $\mu \in \mathbb{R}$, given by $(M_\mu f)(s) := e^{i\mu s} f(s)$ is an isomorphism on $\mathcal{A}(\mathbb{R}, X)$. Since $M_{-\mu}T(t)M_\mu = e^{-i\mu t}T(t)$ for $t \geq 0$, we obtain (i) and (ii). □

We are now in a position to prove the spectral mapping theorem for evolution semigroups on $AP(\mathbb{R}, X)$.

THEOREM 3.6. *Let $\mathcal{U} = \{U(t, s) : t \geq s \in \mathbb{R}\}$ be an a.p. evolution family on the Banach space X such that $\lim_{t\to 0} U(s, s-t)x = x$, $x \in X$, uniformly for $s \in \mathbb{R}$. Let $\mathcal{T}$ be the corresponding evolution semigroup on $AP(\mathbb{R}, X)$ with generator $(G, D(G))$. Then*

$$\sigma(T(t)) \setminus \{0\} = e^{t\sigma(G)}$$

for all $t \geq 0$.

Proof. It suffices to show that if $1 \in A\sigma(T(1))$, then $0 \in \sigma(G)$ (cf. [RS96, Section 1]). So, let $1 \in A\sigma(T(1))$. Then, for all $N \geq 2$ and $c > 0$, there exists a function $f \in AP(\mathbb{R}, X)$ such that $\|f\| = 1$, $\|T(1)^N f - f\| \leq c$ and $\|T(1)^k f\| \leq 2$, $k = 0, 1, \cdots, 2N$. Let $s \in (0, 1]$ such that $\|T(t+N)f - T(N)f\| \leq d$ for $|t| \leq s < 1$. Choose $\gamma \in C_c^1(\mathbb{R}^+)$ (C^1-function with compact support) such that $0 \leq \gamma \leq 1$ and

$\gamma|_{\mathbb{R}^+ \setminus (0, 2N)} = 0,$

$|\gamma'(t)| \leq \frac{2}{N}$ for all $t \in \mathbb{R}^+$,

$\gamma(t) = 1$ for $t \in (N - s, N + s)$.

Choose $t_0 \in \mathbb{R}$ such that $\|f(t_0)\| \geq 1-\epsilon$. Choose $\alpha : \mathbb{R} \to [0,1]$ periodic (of period $2N$) such that $0 \leq \alpha \leq 1$ and

$$\alpha|_{(t_0-N-\frac{s}{2},t_0-N+\frac{s}{2})} \equiv 1,$$

$$\alpha|_{[t_0-2N,t_0-N-s)\cup(t_0-N+s,t_0)} \equiv 0.$$

Let $g := \int_0^\infty \gamma(t)T(t)(\alpha f)dt \in AP(\mathbb{R},X)$. Note that $g \in D(G)$ and $Gg = -\int_0^\infty \gamma'(t)T(t)(\alpha f)dt$ (cf. [EN00, Proposition 1.8]). Thus

$$\begin{aligned} \|Gg\| &= \left\| \int_0^{2N} \gamma'(t)\alpha(\cdot - t)T(t)f dt \right\| \\ &\leq \frac{2}{N} \max_{0\leq t\leq 2N} \|T(t)f\| \left\| \int_0^{2N} \alpha(\cdot - t)dt \right\|. \end{aligned}$$

Now, $\max_{0\leq t\leq 2N} \|T(t)f\| \leq 2C$ where $C := \sup_{0\leq t\leq 1} \|T(t)\|$. Moreover, $\int_0^{2N} \alpha(r-t)dt \leq 2s$ for all $r \in \mathbb{R}$. Hence $\|Gg\| \leq \frac{2}{N} \cdot 2C \cdot 2s = \frac{8Cs}{N}$. On the other hand, $\|g\| \geq \|g(t_0)\|$ where

$$\begin{aligned} g(t_0) &= \int_0^{2N} \gamma(t)\alpha(t_0-t)T(t)f(t_0)dt \\ &= \int_{-N}^{N} \gamma(t+N)\alpha(t_0-t-N)T(t+N)f(t_0)dt \\ &= \int_{-s}^{s} \gamma(t+N)\alpha(t_0-t-N)T(t+N)f(t_0)dt \\ &= \int_{-s}^{s} \alpha(t_0-t-N)T(t+N)f(t_0)dt \\ &= f(t_0)\int_{-s}^{s} \alpha(t_0-t-N)dt \\ &\quad + (T(N)-Id)f(t_0)\int_{-s}^{s} \alpha(t_0-t-N)dt \\ &\quad + \int_{-s}^{s} \alpha(t_0-t-N)(T(t+N)-T(N))f(t_0)dt \\ &= I_1 + I_2 + I_3. \end{aligned}$$

It follows that

$$\begin{aligned} \|I_1\| &\geq (1-\epsilon)s \\ \|I_2\| &\leq \|(T(N)-Id)f\| \int_{-s}^{s} \alpha(t_0-t-N)dt \leq 2cs \\ \|I_3\| &\leq \int_{-s}^{s} \alpha(t_0-t-N)\|T(t+N)f - T(N)f\|dt \leq 2ds. \end{aligned}$$

So, $\|g(t_0)\| \geq (1-\epsilon)s - 2(c+d)s$ where c, d and ϵ can be choosen arbitrary small. Therefore, $\|g(t_0)\| \geq \frac{s}{2}$ can be achieved and we obtain $\|Gg\| \leq \frac{6}{N}C\|g\|$. □

Note that $C_0(\mathbb{R},X)$ is contained in $AAP_r^+(\mathbb{R},X)$, $W_r(\mathbb{R},X)$, resp. $W_r^+(\mathbb{R},X)$ (cf. [RS89, p. 16, Remark]). This enables us to deduce a spectral mapping theorem for evolution semigroups on $AAP_r^+(\mathbb{R},X)$, $W_r(\mathbb{R},X)$, resp. $W_r^+(\mathbb{R},X)$ from the spectral mapping theorem for evolution semigroups on $C_0(\mathbb{R},X)$ [LMS96, RS96].

THEOREM 3.7. *Let $\mathcal{U} = \{U(t,s) : t \geq s \in \mathbb{R}\}$ be an a.a.p.r., E.-w.a.p.r., resp. E.-w.a.a.p.r. evolution family on the Banach space X such that $\lim_{t\to 0} U(s,s-t)x = x$, $x \in X$, uniformly for $s \in \mathbb{R}$. Let $\mathcal{T}$ be the corresponding evolution semigroup on $AAP_r^+(\mathbb{R},X)$, $W_r(\mathbb{R},X)$, resp. $W_r^+(\mathbb{R},X)$ with generator $(G,D(G))$. Then*

$$\sigma(T(t)) \setminus \{0\} = e^{t\sigma(G)}$$

for all $t \geq 0$.

Proof. Let $\mathcal{F}(\mathbb{R},X)$ be one of the spaces $AAP_r^+(\mathbb{R},X)$, $W_r(\mathbb{R},X)$ and $W_r^+(\mathbb{R},X)$, respectively. It suffices to show that

$$\text{if } 1 \in A\sigma(T(t_0)) \text{ for some } t_0 > 0, \text{ then } 0 \in \sigma(G)$$

(cf. [RS96, Section 1]). So, let $1 \in A\sigma(T(t_0))$ for some $t_0 > 0$. For each $n \in \mathbb{N}$ there exists $f_n \in \mathcal{F}(\mathbb{R},X)$ such that $\|f_n\| = 1$ and $\|f_n - T(kt_0)f_n\| < \frac{1}{2n}$ for all $0 \leq k \leq 2n$ and $k \in \mathbb{N}$. Then

$$\frac{1}{2} < \sup_{s\in\mathbb{R}} \|U(s,s-kt_0)f_n(s-kt_0)\| < \frac{3}{2}$$

for $0 \leq k \leq 2n$. For each $n \in \mathbb{N}$ choose $s_n \in \mathbb{R}$ such that $\|U(s_n, s_n - nt_0)x_n\| \geq \frac{1}{2}$ where $x_n := f_n(s_n - nt_0)$. Let $I_n := [s_n - nt_0, s_n + nt_0]$ and $\tilde{g}_n := \chi_{I_n} U(\cdot, s_n - nt_0)x_n$. Thus there is a positive constant c such that

$$\frac{1}{2} \leq \sup_{s\in\mathbb{R}} \|\tilde{g}_n(s)\| \leq c \text{ for all } n \in \mathbb{N}.$$

Choose now $\alpha_n \in C^1(\mathbb{R},[0,1])$ such that $\alpha_n(s_n) = 1$, $\operatorname{supp}\alpha_n \subseteq I_n$ and $\|\alpha_n'\| \leq \frac{2}{nt_0}$. Define

$$g_n := \alpha_n \tilde{g}_n = \alpha_n(\cdot)U(\cdot, s_n - nt_0)x_n \in C_0(\mathbb{R},X).$$

Then $\frac{1}{2} \leq \|g_n\| \leq c$ for all $n \in \mathbb{N}$. Let $\mathcal{T}_0$ be the corresponding evolution semigroup on $C_0(\mathbb{R},X)$ with generator G_0. Then $g_n \in D(G_0)$ and $G_0 g_n = -\alpha_n' \tilde{g}_n$ [RS96, Proposition 2.2]. In particular, $\|G_0 g_n\| \leq \frac{2c}{nt_0}$, and hence 0 is an approximate eigenvalue of G_0.

To conclude the proof, note that G_0 is the part of G in $C_0(\mathbb{R},X)$. Therefore 0 is also an approximate eigenvalue of G. □

4. Hyperbolicity of evolution families satisfying almost periodicity properties. Hyperbolic evolution families have already been investigated by several authors (see e.g. [CL99, EN00, LMS96, LMS94, LR95, Rau94a, RS94, RS96] and the literature cited therein). Nevertheless, their results did not take into account the particularities one has when dealing with q-p. or **ap** evolution families. In the following, we characterize hyperbolicity of q-p. evolution families by means of the monodromy operator and the evolution semigroup on $P_q(\mathbb{R}, X)$. Moreover, a similar result is obtained for **ap** evolution families.

The following lemma is the essential step to show that the projection corresponding to a hyperbolic evolution semigroup is a multiplication operator (cf. [Rau94a, Lemma 4] and Section 2).

LEMMA 4.1. *(i) Let $\mathcal{T}$ be a hyperbolic evolution semigroup on $P_q(\mathbb{R}, X)$ with projection $\mathcal{P}$. Then $\varphi\mathcal{P}f = \mathcal{P}\varphi f$ for all $\varphi \in P_q(\mathbb{R})$ and $f \in P_q(\mathbb{R}, X)$.*

(ii) Let $\mathcal{F}$ stand for AP, AAP^+, W and $W^{(+)}$, respectively. Let $\mathcal{T}$ be a hyperbolic evolution semigroup on $\mathcal{F}_r(\mathbb{R}, X)$ with projection $\mathcal{P}$. Then $\varphi\mathcal{P}f = \mathcal{P}\varphi f$ for all $\varphi \in \mathcal{F}(\mathbb{R})$ and $f \in \mathcal{F}_r(\mathbb{R}, X)$.

Proof. In order to show (i), take in the proof of [Rau94a, Lemma 4] $P_q(\mathbb{R}, X)$, resp. $P_q(\mathbb{R})$ instead of $C_0(\mathbb{R}, X)$, resp. $C_b(\mathbb{R})$. We mentioned already (see Theorem 2.5 and Theorem 2.7) that $\varphi f \in \mathcal{F}_r(\mathbb{R}, X)$ whenever $f \in \mathcal{F}_r(\mathbb{R}, X)$ and $\varphi \in \mathcal{F}(\mathbb{R})$. Taking now again in [Rau94a, Lemma 4] $\mathcal{F}_r(\mathbb{R}, X)$, resp. $\mathcal{F}(\mathbb{R})$ instead of $C_0(\mathbb{R}, X)$, resp. $C_b(\mathbb{R})$, assertion (ii) follows. □

If $\mathcal{U}$ is a periodic evolution family which solves the non-autonomous Cauchy problem (2) for bounded linear operators $A(t)$, then it is known from [DK74, V, Theorem 2.1] that $\sigma(V) \cap \Gamma = \emptyset$ is equivalent to the fact that $\mathcal{U}$ is hyperbolic. We refer to [DK92, Hen81] for partial generalizations of this to unbounded operators $A(t)$. Rau showed the above equivalence for general q-p. evolution families which are invertible [Rau94a].

THEOREM 4.2. *Let $\mathcal{U} = \{U(t,s) : t \geq s \in \mathbb{R}\}$ be a q-p. evolution family on the Banach space X, and let $\mathcal{T}$ be the associated evolution semigroup on $P_q(\mathbb{R}, X)$ with generator G. Then the following assertions are equivalent.*

(i) $\mathcal{U}$ is hyperbolic.

(ii) $\sigma(V) \cap \Gamma = \emptyset$.

(iii) $\mathcal{T}$ is hyperbolic.

(iv) $\mathcal{U}$ is hyperbolic with q-p. projections.

(v) There exists $\lambda \in i\mathbb{R}$ such that $\left[\lambda, \lambda + \frac{2\pi i}{q}\right) \subset \rho(G)$.

Proof. We show (i)⇒(ii)⇒(iii)⇒(iv) and (iii)⇔(v). The implication (iv)⇒(i) is obvious.

(i)⇒(ii): Let $\mathcal{U} = \{U(t,s) : t \geq s \in \mathbb{R}\}$ be hyperbolic. Then the corresponding evolution semigroup $\mathcal{T}_0$ on $C_0(\mathbb{R}, X)$ is hyperbolic [RS94, Theorem 1.5], i.e., $\Gamma \subseteq \rho(T_0(q))$. Since $\Gamma\sigma(V) \setminus \{0\} = \sigma(T_0(q)) \setminus \{0\}$ [Rau94a, Proposition 12], we obtain $\sigma(V) \cap \Gamma = \emptyset$.

(ii)⇒(iii): From $\sigma(V) \cap \Gamma = \emptyset$, it follows that $\sigma(T(q)) \cap \Gamma = \emptyset$ (Proposition 3.1(iii)). Therefore, the evolution semigroup $\mathcal{T}$ is hyperbolic.

(iii)⇒(iv): Let $\mathcal{T}$ be hyperbolic with projection $\mathcal{P}$. Then $\varphi \mathcal{P} f = \mathcal{P} \varphi f$ for all $\varphi \in P_q(\mathbb{R})$ and $f \in P_q(\mathbb{R}, X)$ (Lemma 4.1(i)). So, from Theorem 2.4, we see that $\mathcal{P}$ is a multiplication operator on $P_q(\mathbb{R}, X)$. Therefore, there are operators $P(t)$, $t \in \mathbb{R}$, such that $\mathcal{P} = P(\cdot) \in P_q(\mathbb{R}, \mathcal{L}_s(X))$. Furthermore, $P(t)$, $t \in \mathbb{R}$, is the desired family of projections for $\mathcal{U}$ (compare the proof of Theorem 1.5(2)⇒(1), case $p = \infty$, in [RS94]).

(iii)⇔(v): This follows from Propositions 3.1 and 3.2. □

We conclude with the following characterization of hyperbolic evolution semigroups on $\mathcal{A}(\mathbb{R}, X)$.

THEOREM 4.3. *Let $\mathcal{U} = \{U(t,s) : t \geq s \in \mathbb{R}\}$ be an* **ap** *evolution family on the Banach space X such that*

$$\lim_{t \to 0} U(s, s-t)x = x, \; x \in X,$$

uniformly for $s \in \mathbb{R}$. Let $\mathcal{T}$ be the associated evolution semigroup on $\mathcal{A}(\mathbb{R}, X)$ with generator G. Then the following assertions are equivalent.

(i) $\mathcal{T}$ is hyperbolic.

(ii) $\mathcal{U}$ is hyperbolic with projections $P(t)$ and $Q(t) = Id - P(t)$ such that

$$R_t f : \mathbb{R} \to X : s \mapsto U_Q(s+t, s)^{-1} Q(s+t) f(s+t)$$

is an **ap** *function for all $f \in \mathcal{A}(\mathbb{R}, X)$ and $t \geq 0$.*

(iii) $0 \in \rho(G)$.

Proof. (i)⇒(ii): Let $\mathcal{T}$ be hyperbolic with projection $\mathcal{P}$. Then $\varphi \mathcal{P} f = \mathcal{P} \varphi f$ for all $\varphi \in \mathcal{A}(\mathbb{R})$ and $f \in \mathcal{A}(\mathbb{R}, X)$ (Lemma 4.1(ii)). There are operators $P(t)$, $t \in \mathbb{R}$, such that $\mathcal{P} = P(\cdot) \in \mathcal{A}(\mathbb{R}, \mathcal{L}_s(X))$. At this stage, the arguments of Theorem 1.5, case $p = \infty$, in [RS94] show that $\mathcal{U}$ is hyperbolic with projections $P(t)$. Furthermore, for $\mathcal{Q} = Id - \mathcal{P}$,

$$(\mathcal{Q} f)(s+t) = \left(T_{\mathcal{Q}}(t) \mathcal{Q} T_{\mathcal{Q}}(t)^{-1} \mathcal{Q} f\right)(s+t) = U_Q(s+t, s) Q(s) \left(T_{\mathcal{Q}}(t)^{-1} \mathcal{Q} f\right)(s)$$

for all $t \geq 0$, $s \in \mathbb{R}$ and $f \in \mathcal{A}(\mathbb{R}, X)$. Therefore,

$$R_t f = T_{\mathcal{Q}}(t)^{-1} \mathcal{Q} f \in \mathcal{A}(\mathbb{R}, X).$$

(ii)⇒(i): If $\mathcal{U}$ is hyperbolic with projections $P(t)$ and $R_t f \in \mathcal{A}(\mathbb{R}, X)$ for all $f \in \mathcal{A}(\mathbb{R}, X)$ and $t \geq 0$, then

$$P(\cdot)x = (Id - R_0)\mathbf{1} \otimes x \in \mathcal{A}(\mathbb{R}, X)$$

for all $x \in X$, and $\mathcal{P} = P(\cdot)$ is a bounded projection on $\mathcal{A}(\mathbb{R}, X)$ which commutes with $T(t)$, $t \geq 0$. Let $\mathcal{Q} := Id - \mathcal{P}$. Since $T_{\mathcal{P}}(t)\mathcal{P}f = U(\cdot, \cdot - t)P(\cdot - t)f(\cdot - t)$ and $T_{\mathcal{Q}}(t)^{-1}\mathcal{Q}f = R_t f$ whenever $f \in \mathcal{A}(\mathbb{R}, X)$, we obtain the hyperbolicity of $\mathcal{T}$.

The spectral mapping theorem (Theorem 3.6 and Theorem 3.7) and Proposition 3.5 finally show the equivalence of (i) and (iii). □

REMARK 4.4. Let $\mathcal{U} = \{U(t,s) : t \geq s \in \mathbb{R}\}$ be an **ap** evolution family on the Banach space X such that $\lim_{t\to 0} U(s, s-t)x = x$, $x \in X$, uniformly for $s \in \mathbb{R}$. Let $\mathcal{T}$ be the associated evolution semigroup on $\mathcal{A}(\mathbb{R}, X)$ with generator G. It is open whether hyperbolicity of $\mathcal{U}$ is equivalent to the hyperbolicity of $\mathcal{T}$. In particular, we do not know whether the hyperbolicity of $\mathcal{U}$ already implies $P(\cdot) \in \mathcal{A}(\mathbb{R}, \mathcal{L}_s(X))$.[1]

5. Almost periodicity of mild solutions of inhomogenous non-autonomous Cauchy problems. Let $\mathcal{U} = \{U(t,s) : t \geq s \in \mathbb{R}\}$ be an evolution family on the Banach space X and let $\mathcal{F} = \mathcal{F}(\mathbb{R}, X)$ be one of the spaces $P_q(\mathbb{R}, X)$ and $\mathcal{A}(\mathbb{R}, X)$. Our aim is the discussion of the following property.
($\mathcal{F}$) *For every $f \in \mathcal{F}$ there exists a unique solution $u \in \mathcal{F}$ of*

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\tau)f(\tau)d\tau, \; t \geq s \in \mathbb{R}.$$

In the finite dimensional case, A.D. Maĭzel [Maĭ54] showed that boundedness of the above function u is connected with hyperbolicity of $\mathcal{U}$, provided that f is bounded and uniformly continuous. His method goes back to a classical work of O. Perron [Per30]. For general Banach spaces X and $\mathcal{F} = C_0(\mathbb{R}, X)$, $C_b(\mathbb{R}, X)$, $L^p(\mathbb{R}, X)$, $1 \leq p < \infty$, it is known that property ($\mathcal{F}$) characterizes hyperbolicity of $\mathcal{U}$ (see [LRS98] and the literature cited therein). Results in this direction can also be found in [CL99, EN00, Cop78, DK74, Prü84].

We show in which way property ($\mathcal{F}$) is connected to hyperbolicity of an evolution family $\mathcal{U}$, assuming that $\mathcal{U}$ has almost periodicity properties. It turns out that, in the q-p. case and for $\mathcal{F} = P_q(\mathbb{R}, X)$, property ($\mathcal{F}$) can be characterized by $1 \in \rho(V)$. In the **ap** case and for $\mathcal{F} = \mathcal{A}(\mathbb{R}, X)$, property ($\mathcal{F}$) is equivalent to the hyperbolicity of $\mathcal{U}$ in the sense of Theorem 4.3.

The following lemma represents the key for connecting hyperbolicity of an evolution family with property ($\mathcal{F}$). For a similar result concerning evolution families on the half-line, we refer to [MRS98, Lemma 1.1].

LEMMA 5.1. *Let $\mathcal{T}$ be an evolution semigroup on $\mathcal{F} = P_q(\mathbb{R}, X)$ resp. $\mathcal{A}(\mathbb{R}, X)$ with generator $(G, D(G))$. Let $\mathcal{U}$ be the evolution family corresponding to $\mathcal{T}$. Consider $u, f \in \mathcal{F}$. Then the following assertions are equivalent.*

(i) $u \in D(G)$ *and* $Gu = -f$.

(ii) $u(t) = U(t,s)u(s) + \int_s^t U(t,\tau)f(\tau)d\tau$ *for* $t \geq s$.

[1] Recently, in [MS02], this question has been discussed for inhomogeneous evolution equations (3) assuming that the operators $A(t)$ satisfy the Acquistapace-Terreni conditions.

Proof. (i)⇒(ii): Let $u \in D(G)$ and $Gu = -f$. Since $\mathcal{T}$ is a $\mathcal{C}_0$-semigroup, we obtain

$$\begin{aligned}
-U(t,s)u(s) + u(t) &= -(T(t-s)u)(t) + u(t) = -\int_0^{t-s} (T(\tau)Gu)(t)d\tau \\
&= \int_0^{t-s} (T(\tau)f)(t)d\tau = \int_0^{t-s} U(t,t-\tau)f(t-\tau)d\tau \\
&= \int_s^t U(t,\tau)f(\tau)d\tau
\end{aligned}$$

for $t \geq s$.

(ii)⇒(i): Note that

$$\frac{-T(t)u + u}{t} = \frac{1}{t}\int_0^t T(\tau)f d\tau \stackrel{t\to 0}{\longrightarrow} f$$

for $u, f \in \mathcal{F}$ satisfying (ii). □

The main consequence of Lemma 5.1 is the fact that property $(\mathcal{F})$ can be characterized by the invertibility of the generator G of the evolution semigroup $\mathcal{T}$ on $\mathcal{F}$. Together with Theorem 4.3, we obtain the following result for $\mathcal{F} = \mathcal{A}(\mathbb{R}, X)$.

THEOREM 5.2. *Let $\mathcal{T}$ be an evolution semigroup on $\mathcal{F} = \mathcal{A}(\mathbb{R}, X)$ with generator $(G, D(G))$. Let $\mathcal{U} = \{U(t,s) : t \geq s \in \mathbb{R}\}$ be the evolution family corresponding to $\mathcal{T}$. Then the following assertions are equivalent.*

(i) Property $(\mathcal{F})$ holds.

(ii) $\mathcal{U}$ is hyperbolic with projections $P(t)$ and $R_t f \in \mathcal{F}$ for all $f \in \mathcal{F}$ and $t \geq 0$.

(ii) $0 \in \rho(G)$.

Finally, in the q-p. case, we establish, in addition, a spectral characterization of property $(\mathcal{F})$ using the monodromy operator.

THEOREM 5.3. *Let $\mathcal{T}$ be an evolution semigroup on $\mathcal{F} = P_q(\mathbb{R}, X)$ with generator $(G, D(G))$. Let $\mathcal{U} = \{U(t,s) : t \geq s \in \mathbb{R}\}$ be the evolution family corresponding to $\mathcal{T}$. Then the following assertions are equivalent.*

(i) Property $(\mathcal{F})$ holds.

(ii) $1 \in \rho(V)$.

(iii) $0 \in \rho(G)$.

Proof. We have to show that

$$0 \in \rho(G) \Leftrightarrow 1 \in \rho(V).$$

However, this follows immediately from Propositions 3.1 and 3.2. □

We close this section with two examples.

EXAMPLE 5.4. Let Ω be an open bounded set of $\mathbb{R}^n$ with C^2 boundary $\partial\Omega$, and let ν be the unit vector normal to $\partial\Omega$. Consider

$$u_t(t,x) - \sum_{i,j=1}^{n} D_i\left(a_{i,j}(t,x)D_j u(t,x)\right) + u(t,x) = f(t,x),\ (t,x) \in \mathbb{R}\times\Omega,$$

$$\sum_{i,j=1}^{n} a_{i,j}(t,x)(D_j u(t,x))\nu_i(x) = 0,\ (t,x) \in \mathbb{R}\times\partial\Omega.$$

Suppose that there are $\alpha, \eta > 0$ such that

$$a_{i,j} \in C^{\alpha+1/2}\left(\mathbb{R}, C^0\left(\overline{\Omega}\right)\right) \cap C^{\alpha}\left(\mathbb{R}, C^1\left(\overline{\Omega}\right)\right),$$

$$\operatorname{Re}\left(\sum_{i,j=1}^{n} a_{i,j}(t,x)\overline{\xi_i}\xi\right) \geq \eta|\xi|^2 \text{ for } (t,x) \in \mathbb{R}\times\overline{\Omega},\ \xi \in \mathbb{C}^n.$$

Suppose further that there exists a real number $q > 0$ such that

$$a_{i,j}(t+q,x) = a_{i,j}(t,x),\ (t,x) \in \mathbb{R}\times\overline{\Omega}.$$

To obtain an abstract Cauchy problem (2), we rewrite this parabolic problem by setting $X := L^2(\Omega)$, $f : \mathbb{R} \to X : t \mapsto f(t,\cdot)$ and

$$D(A(t)) := \left\{ v \in W^{2,2}(\Omega) : \sum_{i,j=1}^{n} a_{i,j}(t,x)(D_j v)\nu_i(x) = 0,\ (t,x) \in \mathbb{R}\times\partial\Omega \right\}$$

$$A(t)v := \sum_{i,j=1}^{n} D_i\left(a_{i,j}(t,x)D_j v\right) + v,\ v \in D(A(t)).$$

It is shown in [Fuh91] that the above assumptions lead to a q-p. evolution family $\mathcal{U}$ induced by the solutions of

$$\dot{u}(t) = A(t)u(t),\ t \in \mathbb{R}.$$

Moreover, there exists $0 < \rho < 1$ such that $\sigma(V) \cap \{\lambda \in \mathbb{C} : |\lambda| = \rho\} = \emptyset$ [Fuh91, Section 7]. Choose $\gamma \in \mathbb{R}$ such that $\rho = e^{-\gamma q}$. Consider the q-p. evolution family $\tilde{\mathcal{U}} = \left\{e^{\gamma(t-s)}U(t,s) : t \geq s\right\}$. Then $\sigma(\tilde{V}) \cap \Gamma = \emptyset$ and, for every $f \in C_b(\mathbb{R}, X)$ such that $(t \mapsto e^{\gamma t}f(t)) \in \mathcal{A}(\mathbb{R}, X)$, there is a unique solution u of

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\tau)f(\tau)d\tau,\ t \geq s$$

such that $(t \mapsto e^{\gamma t}u(t)) \in \mathcal{A}(\mathbb{R}, X)$. This follows from Theorems 4.2 and 5.2 applied to $\tilde{\mathcal{U}}$.

Example 5.5. Let A_0 be a sectorial operator in X and $0 \leq \alpha < 1$. Let X^α be the space induced by the fractional power A_0^α of A_0 (see [Hen81, p. 29]). Assume that $t \mapsto A(t) - A_0 : \mathbb{R} \to \mathcal{L}(X^\alpha, X)$ is a locally Hölder continuous a.p. function. Then the corresponding evolution family $\mathcal{U}$ on X fulfils the assumptions of Theorem 5.2 (compare [Hen81, p. 240] and [RRSV00] for the exponential boundedness of $\mathcal{U}$). Let $\mathcal{F} = AAP_r^+(\mathbb{R}, X)$. The following assertions are equivalent.

(i) Property ($\mathcal{F}$) holds.

(ii) $\mathcal{U}$ is hyperbolic.

Proof. Let $\mathcal{U} = \{U(t,s) : t \geq s \in \mathbb{R}\}$ be hyperbolic with projections $P(t)$. In [Hen81, p. 240] it is shown that under the above conditions we have $R_t f \in AAP_r^+(\mathbb{R}, X)$ for all $f \in AAP_r^+(\mathbb{R}, X)$, $t \geq 0$. The assertion now follows from Theorem 5.2. □

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