The spectral mapping theorem for an evolution semigroup on the space of almost periodic functions

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Abstract

We show that the spectral mapping theorem holds for an evolution semigroup on the space of almost periodic functions

Key words: Spectral mapping theorem, evolution family, evolution semigroup, almost periodic.

1 Introduction

In [Hut97, Remark 2.3.6] the author states that it is not known, whether the spectral mapping theorem holds for evolution semigroups on the space $AP(\mathbb{R}, X)$ of almost periodic functions from $\mathbb{R}$ to a Banach space X. The purpose of this paper is to give an affirmative answer to this open question.

Given a (strongly continuous) almost periodic evolution family $\mathcal{U} = \{U(t,s) : t \geq s \in \mathbb{R}\}$ of bounded linear operators on a Banach space X, i.e.,

$$U(t + \cdot, s + \cdot)x : \mathbb{R} \to X$$

are almost periodic functions for all $t \geq s$, then

$$T(t)f := U(\cdot, \cdot - t)f(\cdot - t),\ t \geq 0,\ f \in AP(\mathbb{R}, X)$$

defines a $\mathcal{C}_0$-semigroup $\mathcal{T} = \{T(t) : t \geq 0\}$ on $AP(\mathbb{R}, X)$, the so called *evolution semigroup*, provided that

$$\lim_{t \to 0} U(s, s-t)x = x$$

for all $x \in X$, uniformly for $s \in \mathbb{R}$ (see [Hut96, Hut97] and the literature cited therein).

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2 Proof of the spectral mapping theorem

The spectral mapping theorem for evolution semigroups on $AP(\mathbb{R}, X)$ has the following formulation.

Theorem 2.1 *Let $\mathcal{U} = \{U(t,s) : t \geq s \in \mathbb{R}\}$ be an almost periodic evolution family on the Banach space X such that $\lim_{t\to 0} U(s, s-t)x = x$, $x \in X$, uniformly for $s \in \mathbb{R}$. Let $\mathcal{T}$ be the corresponding evolution semigroup on $AP(\mathbb{R}, X)$ with generator $(G, D(G))$. Then*

$$\sigma(T(t)) \setminus \{0\} = e^{t\sigma(G)}$$

for all $t \geq 0$.

Proof. It suffices to show that if $1 \in A\sigma(T(1))$, then $0 \in \sigma(G)$ (cf. [RS96, Section 1]). So, let $1 \in A\sigma(T(1))$. Then, for all $N \geq 2$ and $c > 0$, there exists a function $f \in AP(\mathbb{R}, X)$ such that $\|f\| = 1$, $\|T(1)^N f - f\| \leq c$ and $\|T(1)^k f\| \leq 2$, $k = 0, 1, \cdots, 2N$. Let $s \in (0,1]$ such that $\|T(t+N)f - T(N)f\| \leq d$ for $|t| \leq s < 1$. Choose $\gamma \in C_c^1(\mathbb{R}^+)$ (with compact support) such that $0 \leq \gamma \leq 1$ and

$$\gamma|_{\mathbb{R}^+\setminus(0,2N)} = 0,$$

$$|\gamma'(t)| \leq \tfrac{2}{N} \text{ for all } t \in \mathbb{R}^+,$$

$$\gamma(t) = 1 \text{ for } t \in (N-s, N+s).$$

Choose $t_0 \in \mathbb{R}$ such that $\|f(t_0)\| \geq 1 - \epsilon$. Choose $\alpha : \mathbb{R} \to [0,1]$ periodic (of period $2N$) such that $0 \leq \alpha \leq 1$ and

$$\alpha|_{(t_0-N-\frac{s}{2}, t_0-N+\frac{s}{2})} \equiv 1,$$

$$\alpha|_{[t_0-2N, t_0-N-s)\cup(t_0-N+s, t_0)} \equiv 0.$$

Let $g := \int_0^\infty \gamma(t)T(t)(\alpha f)dt \in AP(\mathbb{R}, X)$. Note that $g \in D(G)$ and $Gg = -\int_0^\infty \gamma'(t)T(t)(\alpha f)dt$ (cf. [EN00, Proposition 1.8]). Thus

$$\begin{aligned} \|Gg\| &= \left\| \int_0^{2N} \gamma'(t)\alpha(\cdot - t)T(t)f\,dt \right\| \\ &\leq \frac{2}{N} \max_{0\leq t\leq 2N} \|T(t)f\| \left\| \int_0^{2N} \alpha(\cdot - t)dt \right\|. \end{aligned}$$

Now, $\max_{0\leq t\leq 2N} \|T(t)f\| \leq 2C$ where $C := \sup_{0\leq t\leq 1} \|T(t)\|$. Moreover, $\int_0^{2N} \alpha(r-t)dt \leq 2s$ for all $r \in \mathbb{R}$. Hence $\|Gg\| \leq \frac{2}{N} \cdot 2C \cdot 2s = \frac{8Cs}{N}$. On the other hand, $\|g\| \geq \|g(t_0)\|$ where

$$g(t_0) = \int_0^{2N} \gamma(t)\alpha(t_0 - t)T(t)f(t_0)dt$$

$$
\begin{aligned}
&= \int_{-N}^{N} \gamma(t+N)\alpha(t_0-t-N)T(t+N)f(t_0)dt \\
&= \int_{-s}^{s} \gamma(t+N)\alpha(t_0-t-N)T(t+N)f(t_0)dt \\
&= \int_{-s}^{s} \alpha(t_0-t-N)T(t+N)f(t_0)dt \\
&= f(t_0)\int_{-s}^{s} \alpha(t_0-t-N)dt \\
&\quad + (T(N)-Id)f(t_0)\int_{-s}^{s} \alpha(t_0-t-N)dt \\
&\quad + \int_{-s}^{s} \alpha(t_0-t-N)(T(t+N)-T(N))f(t_0)dt \\
&= I_1+I_2+I_3.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|I_1\| &\geq (1-\epsilon)s \\
\|I_2\| &\leq \|(T(N)-Id)f\|\int_{-s}^{s} \alpha(t_0-t-N)dt \leq 2cs \\
\|I_3\| &\leq \int_{-s}^{s} \alpha(t_0-t-N)\|T(t+N)f-T(N)f\|dt \leq 2ds.
\end{aligned}
$$

So, $\|g(t_0)\| \geq (1-\epsilon)s-2(c+d)s$ where c,d and ϵ can be choosen arbitrary small. Therefore, $\|g(t_0)\| \geq \frac{s}{2}$ can be achieved and we obtain $\|Gg\| \leq \frac{6}{N}C\|g\|$. □

3 Almost periodicity of mild solutions of inhomogenous non-autonomous Cauchy problems

Let $\mathcal{U} = \{U(t,s) : t \geq s \in \mathbb{R}\}$ be an evolution family on the Banach space X and let $\mathcal{F} = AP(\mathbb{R}, X)$. Our aim is the discussion of the following property.

($\mathcal{F}$) *For every $f \in \mathcal{F}$ there exists a unique solution $u \in \mathcal{F}$ of*

$$
u(t) = U(t,s)u(s) + \int_s^t U(t,\tau)f(\tau)d\tau,\ t \geq s \in \mathbb{R}.
$$

In the finite dimensional case, A.D. Maĭzel [Maĭ54] showed that boundedness of the above function u is connected with hyperbolicity of $\mathcal{U}$, provided that f is bounded and uniformly continuous. His method goes back to a classical work of O. Perron [Per30]. For general Banach spaces X and $\mathcal{F} = C_0(\mathbb{R}, X),\ C_b(\mathbb{R}, X),\ L^p(\mathbb{R}, X),\ 1 \leq p < \infty$, it is known that property ($\mathcal{F}$) characterizes hyperbolicity of $\mathcal{U}$ (see [LRS96] and the literature cited therein, and also [Cop78, DK74, Prü84, CL99, EN00]).

A characterization of property $\mathcal{F}$ for evolution families being asymptotically almost periodic as well as weakly almost periodic in the sense of Eberlein can be found in [Hut96, Hut97]. We obtain our following Theorem by replacing in [Hut96, Section 5] the space $AAP_r^+(\mathbb{R}, X)$ of asymptotically almost periodic functions with relatively compact range by the space $AP(\mathbb{R}, X)$. The spectral characterization $(i) \Leftrightarrow (ii)$ follows from Theorem 2.1.

Theorem 3.1 *Let $\mathcal{T}$ be an evolution semigroup on $\mathcal{F} = AP(\mathbb{R}, X)$, with generator $(G, D(G))$. Let $\mathcal{U} = \{U(t,s) : t \geq s \in \mathbb{R}\}$ be the evolution family corresponding to $\mathcal{T}$. Then the following assertions are equivalent.*

(i) Property $(\mathcal{F})$ holds.

(ii) $\mathcal{U}$ is hyperbolic with projections $P(t)$ and $Q(t) = Id - P(t)$ such that

$$R_t f : \mathbb{R} \to X : s \mapsto U_Q(s+t, s)^{-1} Q(s+t) f(s+t)$$

is an almost periodic function for all $f \in AP(\mathbb{R}, X)$ and $t \geq 0$.

(ii) $0 \in \rho(G)$, where $\rho(G)$ is the resolvent set of G.

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